

# Interim Annual Technical Report

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## Mixed $H_2/H_\infty$ -Control With Output Feedback Compensators Using Parameter Optimization

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# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Norm Definitions and Preliminaries</b>	<b>4</b>
2.1	$H_2$ and $H_\infty$ -Norms and their Properties . . . . .	4
2.2	$H_2$ -Norm of a System . . . . .	5
2.3	$H_\infty$ -Norm of a System and Some of its Properties . . . . .	6
2.4	A Time-Domain Characterization of the $H_\infty$ -Norm . . . . .	8
2.5	Signals with Bounded Power and Bounded Spectrum . . . . .	9
2.6	Some Game Theoretical Results . . . . .	11
<b>3</b>	<b><math>H_\infty</math>-Analysis and Synthesis</b>	<b>13</b>
3.1	Uncertainty Representation and the Small Gain Theorem . . . . .	13
3.2	The General Setup for $H_\infty$ -Analysis and Synthesis . . . . .	14
3.3	Q-parametrization and All Stabilizing Controllers . . . . .	15
3.4	State-Space Formulae for the $H_\infty$ -Bound Problem . . . . .	18
3.4.1	The DGKF Results . . . . .	19
3.4.2	The Zhou and Khargonekar Results . . . . .	21
<b>4</b>	<b>Current Approaches in Mixed <math>H_2/H_\infty</math>-Control</b>	<b>23</b>
4.1	The Bernstein and Haddad Approach . . . . .	23
4.2	The Doyle, Zhou and Bodenheimer Approach . . . . .	27
4.3	The Rotea and Khargonekar Results . . . . .	29
4.3.1	A Bernstein-Haddad Equivalent Setup with State and Static Output Feedback . . . . .	29
4.3.2	A System with Two Inputs and Two Outputs . . . . .	31
4.4	Other Approaches in Mixed $H_2/H_\infty$ -Control . . . . .	33
4.4.1	Mustafa's Entropy Function . . . . .	33
4.4.2	Steinbuch and Bosgra . . . . .	35
4.5	Summary . . . . .	35
<b>5</b>	<b>On the Cost Functional <math>\int_0^{t_f} \{z^T(t)z(t) - \gamma^2 w^T(t)w(t)\} dt</math></b>	<b>35</b>
<b>6</b>	<b>A LQ-Type Cost Function for Mixed <math>H_2/H_\infty</math>-Control Design</b>	<b>42</b>
6.1	The One Disturbance/Two Criterion Case . . . . .	42
6.1.1	Interpretation of the Cost Function and Comments . . . . .	45
6.2	The Two Disturbance/One Criterion Case . . . . .	46
<b>7</b>	<b>Algorithm For The Proposed Mixed <math>H_2/H_\infty</math>-Control Design</b>	<b>48</b>
<b>8</b>	<b>Summary and Concluding Remarks</b>	<b>52</b>
<b>9</b>	<b>Outline of Further Research</b>	<b>52</b>

# 1 Introduction

Among the many possible norm-based optimization methods, the concept of  $H_\infty$ -optimal control has gained enormous attention in the past few years. Attractivity of design methods based on the  $H_\infty$ -norm is due to the fact that this methodology takes, in a direct and effective manner, unmodeled dynamics and other uncertain system parameters into account. Early LQ-based methods tried to account for these uncertainties by introducing, in addition to the real process and sensor noises, fictitious noises into the output and state equations of the linear plant model. This assumption led to the LQG design methodology using Kalman filters as the stochastic counterpart to the deterministic Luenberger observer. In most cases, however, the assumption of random noises entering the system having a certain distribution is not valid.

Here the  $H_\infty$ -framework, based on the Small Gain Theorem and the Youla- (Q-) Parametrization, effectively treats system uncertainties in the control-law synthesis. The  $H_\infty$ -norm of a system can be interpreted as the maximum “gain” of a transfer function in the frequency domain. The Small Gain Theorem then provides a pathway to define robust stability of a system to uncertain inputs in terms of the  $H_\infty$ -norm of this system. Under certain assumptions, the  $H_\infty$ -bound control problem using full-state feedback and the observer-based (full or reduced order)  $H_\infty$ -output feedback problem have been mostly solved. However, when such a solution exists, it is usually not unique. Namely, there exists an infinite number of controllers satisfying the given  $H_\infty$ -bound. The problem of actually minimizing the  $H_\infty$ -norm of a system has proven to be undesirable in many cases as it results in controllers with high gains or large control bandwidth.

A design approach involving a mixed  $H_2/H_\infty$  norm strives to combine the advantages of both methods. The problem is to

- Minimize the  $H_2$ -norm of the closed-loop transfer function and thus minimize the RMS values of the outputs for a system driven by white-noise inputs, and at the same time
- Keep the  $H_\infty$ -norm of the closed-loop system below a certain bound, or minimize this norm to gain robustness and account for uncertain exogenous disturbances that have a certain norm bound.

This advantage motivates research activities toward finding solutions to the mixed  $H_2/H_\infty$ -control problem. In general, a mixed  $H_2/H_\infty$ -control can be termed as ‘LQG with robust stability’, a property that regular LQG cannot provide. The problem of mixed  $H_2/H_\infty$ -control using state feedback or full and reduced order dynamic compensators has been addressed intensively over the past two years. Current research in this field will be discussed with more depth in Chapter 4.

As in LQ-based methods, the  $H_\infty$ -control problem and the mixed  $H_2/H_\infty$ -control problem with controllers of fixed structure and order remains a topic of current research. Some theoretical results have been found and numerical implementation of the solution algorithm remains difficult, mostly relying on approximations and numerical optimization methods. The most frequently used method for low-order controller design consists of first performing a plant order reduction which is then followed by a control-law synthesis using the reduced-order plant model.

The approach developed in our research is based on a finite time cost functional that depicts an  $H_\infty$ -bound control problem in a  $H_2$ -optimization setting. The assumption of a finite-time cost functional is very attractive as it allows the optimization process to start with an initial controller guess that is not necessarily stabilizing. Another motivation for the use of a finite time criterion in the mixed  $H_2/H_\infty$ -control problem as well as the  $H_\infty$ -control problem is the fact that, in steady state, existence of a controller is solely based on the solvability of certain algebraic Riccati equations. If a solution cannot be found, the method would break down with an *unbounded* objective function. The time-domain approach - and in particular using a finite-time horizon - offers more insights into the nature of the problem and provides a uniquely different, non-Riccati based controller design method.

Finally, the nature of our research is toward a practical implementation of the  $H_\infty$ -control algorithms for arbitrary fixed controller structures. We feel strongly that practical application has fallen short of all the existing theory in  $H_\infty$  and mixed  $H_2/H_\infty$ -control. It is necessary to develop algorithms that address practical problems faced by control design engineers and to provide design tools for this purpose in the fields mentioned above. Thus, our attempts will follow a more application oriented line of research rather than a theoretical one.

The goal is to define a time-domain cost function that optimizes the  $H_2$ -norm of a system with an  $H_\infty$ -constraint function. With this cost functional and as  $t_f \rightarrow \infty$ , a necessary and sufficient condition for  $\|H\|_\infty < \gamma$  can be established where  $\gamma$  is a prespecified parameter. Thus, for a finite  $t_f$  the constraint provides 'information' for the  $H_2$ -optimization to yield the desired  $H_\infty$ -bound. It is desired to append a possible  $H_\infty$ -constraint either as a side constraint (in form of an inequality) or, to incorporate the constraint directly into the function on which the mixed  $H_2/H_\infty$ -control is based. The finite-time approach taken is advantageous due to the fact that no stabilizing controller is required in this procedure. Furthermore, for a finite terminal time, all considered functions in the cost functional and their sensitivity to design parameters are well behaved. The optimal controller will then be achieved in the limit of  $t_f \rightarrow \infty$ . In most practical situations, the iterative procedure converges to a nearly steady-state solution when  $t_f$  is relatively large compared to the slowest time constant of the closed-loop modes. The resulting controller is then assumed to be an optimal solution for the steady-state case. Of course, after a controller has been designed, the norm bound as well as stability can be confirmed using the singular value and the eigenvalue analysis of the corresponding closed-loop system. It should be noted that when  $t_f$  is small, this approach does not guarantee stability (in contrast to methods based on the Youla parametrization) or satisfy the norm bound (as in methods based on algebraic Riccati equations).

Organization of this research interim report is as follows. In Chapter 2 we go over the basic norm definitions and cover some preliminaries. In Chapter 3 the  $H_\infty$ -theory leading to the mixed  $H_2/H_\infty$ -problem is reviewed. Chapter 4 covers the most recent advances in mixed  $H_2/H_\infty$ -theory. Chapter 5 investigates an important cost function in the time domain. Chapter 6 addresses our approach to the mixed  $H_2/H_\infty$ -control problem with an appended  $H_\infty$ -bound constraint. The precise cost function for a mixed  $H_2/H_\infty$ -control is defined. Chapter 7 discusses a possible algorithm for the numerical solution of the proposed cost function and the controller design. Finally, the remaining research work is outlined in Chapter 8.

It should be noted that review of the LQ-type  $H_2$ -optimization theory falls short of

the coverage presented here for  $H_\infty$ -theory as  $H_2$ -methods have been widely known. The significant part is the  $H_\infty$ -bound characterization.

## 2 Norm Definitions and Preliminaries

Let  $G := (A, B, C, D)$  denote a linear time invariant system as follows.

$$\mathbf{G} : \begin{cases} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{cases} \quad (1)$$

Unless otherwise stated, the initial condition  $x(0)$  of the system is assumed to be zero. Then the transfer function  $G(s)$  from  $w(s)$  to  $z(s)$  is given by  $G(s) = C(sI - A)^{-1}B + D$ .

### 2.1 $H_2$ and $H_\infty$ -Norms and their Properties

Before the actual norm definitions are stated, we define two important frequency-domain spaces.

- The frequency-domain space  $H_2 = H_2(s, C^{m \times n})$  consists of all matrix functions  $F(s) \in C^{m \times n}$  of a complex parameter  $s$  which are analytic in the open right-half plane (that is  $\text{Re}(s) > 0$ ,  $\text{Re}(\cdot)$  denotes the real part of the argument) and fulfill  $\sup_{s, \text{Re}(s) > 0} \int_{-\infty}^{\infty} F^*(s)F(s)ds < \infty$ . That is  $H_2(s, C^{m \times n})$  contains all asymptotically stable transfer functions  $G(s)$  that are strictly proper ( $D = 0$  in (1)).
- The Hardy space - a frequency-domain space -  $H_\infty = H_\infty(s, C^{m \times n})$  consists of all matrix functions  $F(s) \in C^{m \times n}$  of a complex parameter  $s$  which are analytic in the open right half plane (that is  $\text{Re}(s) > 0$ ) and fulfill  $\bar{\sigma}[F(s)] < \infty \quad \forall \text{Re}(s) > 0$  where  $\bar{\sigma}(\cdot)$  denotes the maximal singular value. That is,  $H_\infty(s, C^{m \times n})$  contains all asymptotically stable transfer functions  $G(s)$ .  
 $L_\infty(L_2)$  contains  $H_\infty(H_2)$  and represent functions  $F(s) \in C^{m \times n}$  that are bounded on the  $j\omega$ -axis and proper (strictly proper), stability is not required for these spaces.
- Besides these frequency-domain spaces the most important normed space in the time domain is  $L_2(\mathbb{R})$  where  $\mathbb{R}$  represents real numbers.  $L_2(\mathbb{R})$  represents all square integrable scalar functions of time  $g(t) \in \mathbb{R}$  with  $\|g\|_2^2 = \int_{-\infty}^{\infty} [g(t)]^2 dt < \infty$ . For simplicity in notation, we will use  $L_2$  to stand for both, the frequency domain as well as the time domain space. The exact domain will be clear from the context where it is used.
- The prefix  $R$  denotes real-rational elements of the according frequency-domain spaces. That is,  $RH_2$  and  $RH_\infty$  denote real rational elements of  $H_2$  and  $H_\infty$  respectively.

With the above definitions the following norms can be derived.

## 2.2 $H_2$ -Norm of a System

**Definition 2.1** Let the system  $G:=(A,B,C,0)$  be stable and strictly proper, then

$$\begin{aligned}\|G\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}[G^*(j\omega)G(j\omega)]d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|G\|_F^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^n \{\sigma_i[G(j\omega)]\}^2 d\omega\end{aligned}\quad (2)$$

where  $\|\cdot\|_F$  denotes the Froebenius norm of a transfer function. Equivalent time-domain definitions can be established using Parseval's theorem. A very important interpretation of the 2-norm of a system is given in terms of a measure of the output response of a system having white noises as input disturbances. A characterization of the 2-norm in these stochastic terms can be put as follows.

**Definition 2.2** Let the system  $G:=(A,B,C,0)$  be asymptotically stable and strictly proper. Let  $w$  be a white-noise disturbance input with unit spectral density  $S_{ww}(j\omega) = I$ . Then

$$\|G\|_2^2 = \lim_{t_f \rightarrow \infty} \mathcal{E}[z^T(t_f)z(t_f)] \quad (3)$$

where  $\mathcal{E}$  represents the expectation operator.

A very convenient way to compute the 2-norm of a transfer function is via the observability grammian  $W_o$  or the controllability grammian  $W_c$  making use of the time-domain definitions of the 2-norm. They are defined as follows.

**Definition 2.3** Consider the system  $G:=(A,B,C,D)$  with  $A$  asymptotically stable,  $(A,B)$  stabilizable and  $(A,C)$  observable, then

$$W_o = \int_0^{\infty} e^{A^T t} C^T C e^{A t} dt \quad (4)$$

$$W_c = \int_0^{\infty} e^{A t} B B^T e^{A^T t} dt \quad (5)$$

Obviously these two matrices can also be computed via a Lyapunov solution. Now the 2-norm of a transfer function can be expressed in terms of  $W_o$  and  $W_c$ .

**Definition 2.4** Let the system  $G:=(A,B,C,0)$  be asymptotically stable and strictly proper, then

$$\|G\|_2^2 = \text{Trace}[B^T W_o B] \quad (6)$$

$$= \text{Trace}[C W_c C^T] \quad (7)$$



## 2.3 $H_\infty$ -Norm of a System and Some of its Properties

For transfer functions  $G(s) \in H_\infty$ , the  $H_\infty$ -norm is defined as follows.

**Definition 2.5** *Let the system  $G:=(A,B,C,D)$  be asymptotically stable, then*

$$\|G\|_\infty^2 = \sup_{s, \operatorname{Re}(s) > 0} \bar{\lambda}[G^*(s)G(s)] \quad (8)$$

where  $G^*(s)$  denotes the conjugate complex transpose of  $G(s)$  and  $\bar{\lambda}(\cdot)$  denotes the maximum eigenvalue of the argument. Unless otherwise stated,  $\|\cdot\|_\infty$  will denote the norm defined on  $H_\infty$  rather than  $L_\infty$ . As the Hardy space defined above is a Banach space, the above definition can alternatively be stated as,

**Definition 2.6** *Let the system  $G:=(A,B,C,D)$  be asymptotically stable, then*

$$\|G\|_\infty^2 = \sup_{\omega} \bar{\lambda}[G^*(j\omega)G(j\omega)], \quad (9)$$

that is, an actual computation has to be performed on the imaginary axis only. A computational algorithm to compute the  $H_\infty$  norm - not exactly but as close to the norm as desired - is due to Boyd [2].

**Definition 2.7** *Let the system  $G:=(A,B,C,D)$  have no poles on the  $j\omega$ -axis and define the Hamiltonian matrix,*

$$M_\gamma = \begin{pmatrix} A + BR^{-1}D^TC & \gamma BR^{-1}B^T \\ -\gamma C^TS^{-1}C & -A^T - C^TDR^{-1}B^T \end{pmatrix} \quad (10)$$

where  $R = (\gamma^2 I - D^TD)$  and  $S = (\gamma^2 I - DD^T)$ .

**Theorem 2.1** ([2]) *Let  $A$  be asymptotically stable and  $\gamma > \bar{\sigma}(D)$ , then  $\|G\|_\infty \geq \gamma \iff M_\gamma$  has at least one eigenvalue on the imaginary axis.*

This theorem provides a convenient way to compute the  $\infty$ -norm of a system by iterating on  $\gamma$ . Starting at a large value for  $\gamma > \bar{\sigma}(D)$  the eigenvalues of the associated matrix  $M_\gamma$  are computed. If  $M_\gamma$  has no purely complex eigenvalue, then the chosen  $\gamma$  is larger than  $\|G\|_\infty$  and we can lower  $\gamma$ . The iteration terminates when the largest  $\gamma$  is found for which  $M_\gamma$  has a purely complex eigenvalue. Note that in the case where  $A$  is not asymptotically stable but has no  $j\omega$ -eigenvalue, the above theorem represents a characterization of the  $L_\infty$ -norm of the same transfer function.

An equivalent theorem expresses the same condition in terms of certain algebraic Riccati equations (ARE).

**Lemma 2.1** ([47] [57]) *Consider a system  $G:=(A,B,C,D)$  with  $A$  asymptotically stable,  $(A,C)$  observable,  $(A,B)$  controllable and  $\gamma > \bar{\sigma}(D)$ , then the following statements are equivalent:*

1.

$$\|G\|_\infty < \gamma \quad (11)$$

2.

$M_\gamma$  has no eigenvalues on the imaginary axis (12)

3. The ARE

$$[A^T + C^T D R^{-1} B^T] X_1 + X_1 [A + B R^{-1} D^T C] + \gamma X_1 B R^{-1} B^T X_1 + \gamma C^T S^{-1} C = 0 \quad (13)$$

has a unique real positive definite solution  $X_1 = X_1^T$  such that  $A + B R^{-1} (D^T C + \gamma B^T X_1)$  is asymptotically stable.

4. The ARE

$$[A^T + C^T D R^{-1} B^T] X_2 + X_2 [A + B R^{-1} D^T C] + \gamma^2 X_2 B R^{-1} B^T X_2 + C^T S^{-1} C = 0 \quad (14)$$

has a unique real positive definite solution  $X_2 = X_2^T$  such that  $A + B R^{-1} (D^T C + \gamma^2 B^T X_2)$  is asymptotically stable.

5. The ARE

$$[A + B R^{-1} D^T C] X_3 + X_3 [A^T + C^T D R^{-1} B^T] + \gamma^2 X_3 C^T S^{-1} C X_3 + B R^{-1} B^T = 0 \quad (15)$$

has a unique real positive definite solution  $X_3 = X_3^T$  such that  $A + (B D^T + \gamma^2 X_3 C^T) S^{-1} C$  is asymptotically stable.

This theorem is an extension of lemma 2.3 in [47] and equation 5.43 in [57] for the proper case rather than the strictly proper case. It can be easily derived using the identities  $X_2 = \gamma^{-1} X_1$ ,  $X_3 = \gamma^{-2} X_1^{-1}$  and  $R^{-1} D^T = D^T S^{-1}$ . Note that the notion of existence of a solution to the above ARE's means that the matrices  $X_1$ ,  $X_2$  and  $X_3$  are bounded; that is, any matrix norm applied to them is finite.

A similar characterization of the  $H_\infty$ -norm was derived by Zhou, Khargonekar and Petersen (see e. g. [21]) in terms of a Riccati inequality (also termed Quadratic Matrix Inequality or QMI by Willems [53]). The QMI represents the fundamental tool for the  $H_\infty$ -design method developed by Zhou, Khargonekar and Petersen.

**Lemma 2.2** ([21], Lemma 2.2) Consider a system  $G := (A, B, C, D)$  with  $A$  asymptotically stable and  $\gamma > \bar{\sigma}(D)$ ,  $(A, B)$  stabilizable and  $(C, A)$  observable, then the following two statements are equivalent:

1.

$$\|G\|_\infty < \gamma \quad (16)$$

2. There exists a real positive definite symmetric matrix  $X_4$  such that

$$[A^T + C^T D R^{-1} B^T] X_4 + X_4 [A + B R^{-1} D^T C] + X_4 B R^{-1} B^T X_4 + C^T (I + D R^{-1} D^T) C < 0 \quad (17)$$

Because this representation is the basis for a very general approach to the  $H_\infty$ -norm bound control problem [46], the proof is briefly outlined here.

The system considered is of the form  $G := (A, B, C, D)$  with  $A$  asymptotically stable. Assume that  $\|G\|_\infty < \gamma$  which implies that  $R$  and  $S$  as defined above are positive definite. Now a unitary transformation  $U$  is applied to the system.

$$\begin{pmatrix} z_N \\ w \end{pmatrix} = U \begin{pmatrix} w_N \\ z \end{pmatrix}$$

where  $z_N$  and  $w_N$  are the new output and input vectors (for a complete discussion of this transformation, see [66]). It can be shown then that this transformation leads to a strictly proper system  $G_N := (A_N, B_N, C_N, 0)$ . Since  $U$  is unitary it is easy to show that  $\|G\|_\infty < \gamma \Leftrightarrow \|G_N\|_\infty < \gamma$ . Applying Lemma 5 from Willems ([53]) the desired result follows immediately using the fact that  $A$  is asymptotically stable and a result from Lyapunov stability theory.

Note the term  $C^T(I + DR^{-1}D^T)C$  can be written as  $C^T(I + DR^{-1}D^T)C = \gamma^2 C^T S^{-1}C$  so that the above inequality can also be stated in the following form,

$$[A^T + C^T DR^{-1}B^T]X_4 + X_4[A + BR^{-1}D^T C] + X_4 BR^{-1}B^T X_4 + \gamma^2 C^T S^{-1}C < 0 \quad (18)$$

From these norm definitions and norm bound characterizations, it can be seen that the whole concept of  $H_\infty$  is set in the frequency domain. However, there are possible time domain characterizations of the  $H_\infty$ -norm that are useful in many respects.

## 2.4 A Time-Domain Characterization of the $H_\infty$ -Norm

**Definition 2.8** Consider a system  $G := (A, B, C, D)$  with  $A$  asymptotically stable,  $x(0) = 0$  and  $w(t) \in L_2$ , then

$$\|H\|_\infty^2 = \sup_{w, \|w\|_2=1} \lim_{t_f \rightarrow \infty} \frac{\int_0^{t_f} z^T(t)z(t)dt}{\int_0^{t_f} w^T(t)w(t)dt} \quad (19)$$

$$= \sup_{w, \|w\|_2=1} \frac{\|z\|_2}{\|w\|_2} \quad (20)$$

This is a definition of an operator norm induced by the 2-norm in the input and output space, other restrictions on  $w(t)$  are possible (see e.g. [65]). Physically this norm can be interpreted as the ratio of  $L_2$ -norms of output vector and input vector (The restriction  $\|w\|_2 = 1$  can always be achieved by scaling as long as  $w(t) \in L_2$ ). Note that the function  $w(t) = \bar{w} \exp(j\omega_0 t)$  would achieve this norm for some  $\omega_0$  and  $\bar{w}$ . However, periodic signals do not have a finite  $L_2$ -norm and are therefore excluded from the above definition. An important observation concerning this time-domain definition - but with a finite horizon time - has been made by Boyd [2],[3]. Namely,

**Theorem 2.2** Consider a system  $G := (A, B, C, D)$  with  $A$  asymptotically stable,  $x(0) = 0$  and  $w(t) \neq 0$ ,  $w(t) \in L_2$ , then  $\forall t_f > 0$ ,

$$\int_0^{t_f} z^T(t)z(t)dt \leq \|H\|_\infty^2 \int_0^{t_f} w^T(t)w(t)dt \quad (21)$$

Importance of the finite-time cost function (20) is not based on the fact that it represents a lower bound for the  $H_\infty$ -norm, but in its relationship to a cost function to be defined later. Finally, shown in this section are a few properties of the  $H_\infty$ -norm that are important for the further development and these are standard results of the operator-norm definition (see e.g. [65]). Consider two transfer functions  $G$  and  $H$ ,

$$\|G\|_\infty \geq 0, \quad \|G\|_\infty = 0 \text{ iff } G=0$$

$$\|\alpha G\|_\infty = |\alpha| \|G\|_\infty$$

$$\|G + H\|_\infty \leq \|G\|_\infty + \|H\|_\infty$$

$$\|GH\|_\infty \leq \|G\|_\infty \|H\|_\infty$$

## 2.5 Signals with Bounded Power and Bounded Spectrum

This section reviews some of the concepts important behind the approach of Doyle, Zhou and Khargonekar to the mixed  $H_2/H_\infty$ -control. The signals considered are time-domain signals that can be vectors in general. For this purpose, let us define the following functions.

**Definition 2.9** *The cross-correlation  $R_{uv}(\tau)$  between two time-domain signals  $u$  and  $v$  is defined as*

$$R_{uv}(\tau) = \lim_{t_f \rightarrow \infty} \frac{1}{2t_f} \int_{-t_f}^{t_f} u(t + \tau) v^T(\tau) dt \quad (22)$$

*if it exists and is finite  $\forall \tau$ .*

The Fourier transform of  $R_{uv}(\tau)$  is called the cross-power spectral density  $S_{uv}(j\omega)$  and is defined as follows.

**Definition 2.10** *The cross-power spectral density  $S_{uv}(j\omega)$  between two time-domain signals  $u$  and  $v$  is defined as*

$$S_{uv}(j\omega) = \int_{-\infty}^{\infty} R_{uv}(\tau) e^{-j\omega\tau} d\tau \quad (23)$$

Remarks:

1. The autocorrelation as well as the power spectral density of a signal  $u(t)$  are defined accordingly.
2. It can be shown that

$$R_{uu}(\tau) = R_{uu}^T(-\tau) \geq 0 \quad (24)$$

$$R_{uv}(\tau) = R_{vu}^T(-\tau) \quad (25)$$

$$S_{uu}(j\omega) = S_{uu}^T(j\omega) \geq 0 \quad (26)$$

$$S_{uv}(j\omega) = S_{vu}^T(j\omega) \quad (27)$$

Now two sets of signals with bounded power and bounded spectrum are defined in the spirit of Doyle, Zhou and Bodenheimer ([14], [15]).

Consider a set of vector-valued functions  $u(t)$  that satisfy

$BP_1$ :  $u(t)$  is finite for all  $t$ ,

$BP_2$ :  $R_{uu}(\tau)$  exists and is finite for all  $\tau$

$BP_3$ :  $S_{uu}(j\omega)$  exists (not necessarily bounded)

Now the following two sets are defined.

**Definition 2.11** ([14]) *The set of vector valued functions  $u(t)$  with bounded power will be denoted by  $\mathcal{P}$ ,*

$\mathcal{P} := \{u(t) : u(t) \text{ satisfies } BP_1, BP_2 \text{ and } BP_3\}$   
with the semi-norm

$$\begin{aligned} \|u\|_{\mathcal{P}}^2 &= \lim_{t_f \rightarrow \infty} \frac{1}{2t_f} \int_{-t_f}^{t_f} u^T(t)u(t)dt \\ &= \text{Trace}\{R_{uu}(0)\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\{S_{uu}(j\omega)\}d\omega \end{aligned} \quad (28)$$

defined on the set  $\mathcal{P}$ .

Similarly the set of all signals with bounded spectrum is defined as follows.

**Definition 2.12** ([14]) *The set of vector-valued functions  $u(t) \in \mathcal{P}$  with bounded spectrum will be denoted by  $\mathcal{S}$ ,*

$\mathcal{S} := \{u(t) : u(t) \in \mathcal{P}, \|S_{uu}\|_{\infty} < \infty\}$   
with the semi-norm

$$\|u\|_{\mathcal{S}}^2 = \|S_{uu}(j\omega)\|_{\infty} \quad (29)$$

defined on the set  $\mathcal{S}$ .

Remarks:

1. Note that all  $L_2$ -signals (in the time domain) have zero power, so do all time limited signals.
2. White noise is not a member of  $\mathcal{S}$  as its auto-correlation is infinite at  $\tau = 0$ . White noise in  $\mathcal{S}$  is comparable to periodic functions in  $L_2$ .
3. Let the prefix  $B$  denote the closed unit ball, that is  $B\mathcal{P}$  denotes the set of signals  $u$  with bounded power and semi power norm  $\|u\|_{\mathcal{P}}^2 \leq 1$ , accordingly for  $B\mathcal{S}$ .
4. Note also that  $\mathcal{S} \subset \mathcal{P}$ .

## 2.6 Some Game Theoretical Results

In this section some important results from game theory are reviewed. They are important for the interpretation of the proposed  $H_2/H_\infty$ -cost function and for the numerical algorithm to compute the desired controller.

Consider a min-max problem of the following form,

$$J(\phi_0, \psi_0) = \inf_{\phi} \sup_{\psi} J(\phi, \psi) \quad (30)$$

where  $\phi_0$  represents the optimal minimizing strategy and  $\psi_0$  the optimal maximizing strategy in this game. Without any assumptions on concavity or convexity, the following results can be obtained,

$$\sup_{\psi} \inf_{\phi} J(\phi, \psi) \leq \inf_{\phi} \sup_{\psi} J(\phi, \psi) \quad (31)$$

This is easily verified and is a standard result in game theory. Another important concept in game theory is that of a 'saddle point strategy' which - in mathematical terms - can be stated as follows.

**Definition 2.13** A pair  $\phi_0$  and  $\psi_0$  is called a saddle point strategy of the inf-sup problem  $\inf_{\phi} \sup_{\psi} J(\phi, \psi)$  if

$$J(\phi_0, \psi) \leq J(\phi_0, \psi_0) \leq J(\phi, \psi_0) \quad \forall \phi \in \Phi \quad \text{and} \quad \forall \psi \in \Psi \quad (32)$$

If a saddle point exists, then the 'inf' and 'sup' operators can be interchanged and 'inf sup' is equivalent to 'sup inf'; but the reverse is not necessarily true. However, in general such a point does not always exist. Various necessary conditions have been established to guarantee the existence of such a point, but most of these conditions assume convexity of  $J(\phi, \psi)$  in  $\phi$  and concavity of  $J(\phi, \psi)$  in  $\psi$  as well as continuity and convex compactness of the spaces  $\Phi$  and  $\Psi$ . In many cases, however, these assumptions are very restrictive and cannot be satisfied so that the existence of a saddle point cannot be guaranteed.

In general if one obtains the solution of a min-max problem via some method, it has to be verified in each case individually whether or not these solutions fulfill the definition of a saddle point as stated in definition 2.13. The importance of a saddle point strategy can be appreciated by looking at it as a true, simultaneous minimization and maximization while min-max or max-min always implies a certain order and thus an advantage/disadvantage to either the minimizer or the maximizer. Other important concepts such as ' $\epsilon$ -optimality' or 'saddle point strategies in equilibrium' are not reviewed here. For a brief review see [33] and [29].

It is recognized that most min-max problems do not have optimal solutions that represent a saddle point. In a paper by Salmon ([45]), he has given an approach to the min-max problem from a different point of view. His objective was to find a solution to the following problem.

**Definition 2.14** ([45]) Let  $J(\phi, \psi)$  be a scalar-valued function that is non-negative, continuous in  $\psi \in \Psi$  and continuous in  $\phi \in \Phi$  where  $\Psi$  and  $\Phi$  represent closed and bounded domains. Let  $\phi^*$  denote the minimizing strategy of  $\phi$  and define the following min-max problem,

$$\max_{\psi \in \Psi} J(\phi^*, \psi) = \min_{\phi \in \Phi} \max_{\psi \in \Psi} J(\phi, \psi) \quad (33)$$

for which he found an iterative algorithm that is guaranteed to converge to  $\min_{\phi \in \Phi} \max_{\psi \in \Psi} J(\phi, \psi)$  using a sequence of successive minimization and maximization steps only (under certain assumptions). His algorithm and some of its properties are stated here without proof, a complete discussion can be found in [45].

#### Assumptions:

1. Continuity requirements:  $J(\phi, \psi)$ ,  $\frac{\delta}{\delta \phi_i} \{J(\phi, \psi)\}$  and  $\frac{\delta}{\delta \psi_j} \{J(\phi, \psi)\}$  are continuous in  $\phi$  and  $\psi$ .
2. For each  $\psi \in \Psi$  there is an optimal  $\phi^o(\psi) \in \Phi$  such that  $J(\phi^o(\psi), \psi) = J^o(\psi)$  where  $J^o(\psi)$  is the minimal value achievable by  $\phi$ . This means that the minimizer can generate the optimal minimizing strategy for any  $\psi$ .
3. Let  $\text{grad}_{\phi} J(\phi, \psi) = 0$  and let the  $\phi, \psi$  satisfying this condition be denoted by  $\bar{\phi}$  and  $\bar{\psi}$ . Then it is assumed that  $J(\bar{\phi}, \bar{\psi}) = J^o(\bar{\psi})$ .

#### Salmon's algorithm:

1. Initialization:  
Choose an initial set  $\Psi_0 = \{\psi_{10}, \psi_{20}, \dots\}$ .
2. The  $n^{\text{th}}$ -iteration:  
Perform:  $\min_{\phi \in \Phi} \{\max_{\psi \in \Psi_{n-1}} J(\phi, \psi)\}$ .  
Let  $\phi^n$  denote this minimizing strategy.  
Define  $J_n^m = \max_{\psi \in \Psi_{n-1}} J(\phi^n, \psi)$ .  
Perform:  $\max_{\psi \in \Psi} \{J(\phi^n, \psi)\}$ .  
Let  $\psi^n$  denote this maximizing strategy.  
Define  $J_n^M = J(\phi^n, \psi^n)$ .  
Define  $\Psi_n = \Psi_{n-1} \cup \{\psi^n\}$ .
3. If  $|J_n^M - J_n^m| < \varepsilon$  then stop  
Otherwise repeat step 2.

#### Remarks:

1. Salmon has shown in [45] that  $J_i^m \leq \min_{\phi \in \Phi} \max_{\psi \in \Psi} J(\phi, \psi) \leq J_i^M$  for all  $i = 1, 2, \dots$
2. If above assumptions are satisfied, then the sequences  $\{J_i^m\}$  and  $\{J_i^M\}$  converge to  $\min_{\phi \in \Phi} \max_{\psi \in \Psi} J(\phi, \psi)$  in a finite number of iterations.
3. If there are no min-max strategies in the interior of the corresponding domains, then the algorithm will converge to min-max strategies on the boundary of these domains.
4. Pure continuity of  $J(\phi, \psi)$  is sufficient to show that  $J_i^m$  is a monotonically increasing function of  $i$ . This is based on the fact that we minimize the objective function over an increasing set  $\Psi_i$ .

5. This algorithm does not guarantee convergence to a saddle point. Instead, it converges to a min-max value as defined in the definition 2.14. Also, the resulting min-max value as well as the resulting min- and max-strategies depend on the choice of the initial set  $\Psi_0$  (if no global minimization and maximization algorithms are applied). If  $J(\phi, \psi)$  is convex/concave in its parameters  $\phi$  and  $\psi$  respectively, then the solution will be unique; if the min-max problem does in addition have a saddle point solution then this algorithm will converge towards this point as in this case min-max equals max-min. If convexity/concavity is not given, this algorithm might converge to local saddle point strategies (if saddle point strategies exist at all).

### 3 $H_\infty$ -Analysis and Synthesis

The whole concept of  $H_\infty$ -analysis is based on the 'Small Gain Theorem' and the representation of a system with its uncertainties in a special form called the Q-parametrization (or Youla parametrization). The synthesis on the other side can be approached via factorization methods in the frequency domain or a game theoretical approach. Before we illustrate this concept, we will review how one can use  $H_\infty$ -norm formulation to include system uncertainties into the analysis and design.

#### 3.1 Uncertainty Representation and the Small Gain Theorem

In terms of notation, this section will follow the notation of [63]. Let  $\Delta_i$  and  $\Delta_o$  denote multiplicative system perturbations in the input and output respectively, and  $\Delta_a$  the additive system uncertainties. The perturbed transfer functions can be represented as

$$G(s) = G_0(s)[I + \Delta_i(s)] \quad (34)$$

or

$$G(s) = [I + \Delta_o(s)]G_0(s) \quad (35)$$

or

$$G(s) = G_0(s) + \Delta_a(s) \quad (36)$$

where  $G_0(s)$  represents the nominal system and the disturbances  $\Delta_{(.)}$  are norm bounded in the  $H_\infty$ -sense; that is,  $\|\Delta_{(.)}\|_\infty \leq \gamma$ . Note that possible uncertainties contained in this representation include parametric model uncertainties, non parametric plant uncertainties (such as transfer function perturbations due to identification errors), or neglected nonlinearities to name a few. Note also that the norm assumption implies that the disturbances are required to be stable, e.g.  $\Delta_{(.)} \in RH_\infty$ . Now we can rewrite this system as follows,

$$z(s) = M(s)w(s) \quad (37)$$

$$w(s) = \Delta(s)z(s) \quad (38)$$

where  $\Delta(s)$  contains all the uncertainties and  $M(s)$  represents the undisturbed system model. At this point, we do have to distinguish between 'structured' and 'unstructured' uncertainties, implying that the  $\Delta(s)$  block has a structure that is known (i.e. structured) or, that



nothing but the norm bound is known and there is no need to define its structure (i.e. unstructured). The first kind of uncertainties leads to  $\mu$ -synthesis while the second type of uncertainties leads to the  $H_\infty$ -framework. Let us assume that  $\|\Delta(s)\|_\infty < 1$  (this can always be achieved by scaling).

Equations (37) and (38) represent a 'feedback' connection of  $M(s)$  and  $\Delta(s)$  to which we now can apply the Small Gain Theorem. For this purpose consider the following system

$$z(s) = M(s)[w(s) + r(s)] \quad (39)$$

$$w(s) = \Delta(s)z(s) \quad (40)$$

where  $r(s)$  represents an auxiliary signal. Then

$$\begin{aligned} \|z(s)\|_2 &= \|M(s)r(s) + M(s)\Delta(s)z(s)\|_2 \\ &\leq \|M(s)r(s)\|_2 + \|M(s)\Delta(s)z(s)\|_2 \\ &\leq \|M(s)\|_\infty \|r(s)\|_2 + \|M(s)\|_\infty \|\Delta(s)\|_\infty \|z(s)\|_2 \end{aligned}$$

Thus, the following identity is true,

$$\|z(s)\|_2 \leq \frac{\|M(s)\|_\infty}{1 - \|M(s)\|_\infty \|\Delta(s)\|_\infty} \|r(s)\|_2$$

Now the Small Gain Theorem can be stated as follows.

**Theorem 3.1** *Consider the feedback system formed by  $M(s)$  and  $\Delta(s)$  with  $M(s) \in H_\infty$  and  $\Delta(s) \in H_\infty$ , then we have the following statement,*

*" $\|M(s)\|_\infty \|\Delta(s)\|_\infty < 1 \Rightarrow$  The closed loop system is stable (of bounded gain)".*

It should be noted that the Small Gain Theorem is valid for other transfer function norms as well since the main result is that the system has 'bounded gain' in the closed-loop feedback configuration. Note also that this theorem does not give an 'if and only if' relation; hence it may be conservative. This conservatism can in fact be removed using the  $\mu$ -framework.

### 3.2 The General Setup for $H_\infty$ -Analysis and Synthesis

The frequency dependence of signals and transfer functions is omitted throughout this section for ease of notation. It has to be kept in mind, however, that all considered transfer functions are real, rational functions of  $s$  (except for matrices in the state-space realization). Consider now a plant description in the frequency domain as follows.

$$\Sigma : \begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \quad (41)$$

Remarks:

- The input vector  $w$  contains the exogenous inputs which include signals such as disturbance inputs or noises acting on the plant from 'outside', fictitious noise inputs to achieve certain design specifications, actuator or sensor noises as well as command inputs.

- The input vector  $u$  is the control input and represents all inputs that can be used to control the plant.
- The output vector  $z$  contains all variables that are to be regulated. That is, it contains outputs that are fed back through a  $\Delta$ -block to the input  $w$ . In general one would like to keep these signals as 'small' as possible. A sample of the elements of  $z$  consists of variables such as errors between a real output and its commanded value, system states, components of the measurement output  $y$  or specific control inputs  $u$ , as well as other linear combination of these signals. Weighting factor can be frequency dependent or pure scaling. Other fictitious  $\Delta$ -blocks between outputs  $z_i$  and  $w$  can be formulated to incorporate robust performance. (Note: Some weighting functions or additional  $\Delta$ -blocks for robust performance may already be imbedded into the above model).
- The output vector  $y$  contains all variables that are measurable and thus accessible to the controller (i.e. sensor outputs).

It is very important to clearly identify the signals used in the optimization model for the  $H_\infty$ -framework. Let us now assume that a state-space representation of the above system  $\Sigma$  is given as follows.

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\ z(t) &= C_1x(t) + D_{11}w(t) + D_{12}u(t) \\ y(t) &= C_2x(t) + D_{21}w(t) + D_{22}u(t) \end{cases} \quad (42)$$

so that a state-space representation of each component  $P_{ij}$  is given by

$$\mathbf{P} : \begin{cases} P_{11} := (A, B_1, C_1, D_{11}) \\ P_{12} := (A, B_2, C_1, D_{12}) \\ P_{21} := (A, B_1, C_2, D_{21}) \\ P_{22} := (A, B_2, C_2, D_{22}) \end{cases} \quad (43)$$

The general objective in the  $H_\infty$ -methodology is to find a stabilizing controller  $K(s)$  with  $u(s) = K(s)y(s)$  that minimizes the  $H_\infty$ -norm of the transfer function  $T_{zw}(s)$  from  $w(s)$  to  $z(s)$  (or to keep this norm below a certain prespecified value for robust stability in the face of the uncertainties  $\Delta$ ).

Note that in general the open-loop system  $P(s)$  may be stable or unstable. In the case of an open-loop system, the first step is to find all the controllers that will stabilize the closed-loop system and then look for one that satisfies the  $H_\infty$ -bound constraint.

It is easily verified that the closed-loop transfer function  $T_{zw}(s)$  with the controller  $K(s)$  in place can be written as

$$T_{zw}(s) = \mathcal{F}(P, K) = P_{11} + P_{12}(I - P_{22}K)^{-1}P_{21} \quad (44)$$

### 3.3 Q-parametrization and All Stabilizing Controllers

A complete treatment of this problem would require extensive use of factorization methods whose details are omitted here for brevity. Only the major steps that lead to the Youla (Q-) parametrization will be outlined. Before we do that, the notion of stabilizability of  $\mathcal{F}(P, K)$  has to be defined.

**Definition 3.1**  $\mathcal{F}(P, K)$  is stabilizable if there is a  $K$  for which  $\mathcal{F}(P, K)$  is internally stable. If such a  $K$  exists we will say that  $K$  stabilizes  $P$ .

For a discussion of internal stability in the sense of Desoer and Chen (see e.g. Francis [62], [63] or [57]). Note that stabilizability of  $P$  does not necessarily means that  $P_{11}$  has to be stable, however, if  $P_{11}$  contains unstable modes then these modes have to be observable through  $y$  and controllable through  $u$  (i.e. the controller has to be able to stabilize these unstable modes). This implies that  $(C_2, A)$  needs to be observable and  $(A, B_2)$  needs to be controllable. These are taken as standard assumptions in the  $H_\infty$ -literature.

The following theorem shows that it suffices to concentrate on  $P_{22}$  only, rather than on the whole  $P$ .

**Lemma 3.1** ([62]) Let  $P_{22} := (A, B_2, C_2, D_{22})$  with  $(A, B_2)$  controllable and  $(A, C_2)$  observable, and furthermore  $K$  and  $P_{22}$  are proper transfer functions. Then  $K$  stabilizes  $P$  iff  $K$  stabilizes  $P_{22}$ .

Before we proceed, let's state some results obtained from factorization, namely right and left coprime factorizations and doubly coprime factorizations.

**Definition 3.2** ([63]) Consider two transfer function matrices  $F, G \in RH_\infty$  where  $F$  and  $G$  have the same number of columns, then  $F$  and  $G$  are right-coprime over  $RH_\infty$  iff there exist  $X, Y \in RH_\infty$  such that

$$XF + YG = I, \quad (45)$$

equivalently, for the left coprime factorization,

**Definition 3.3** ([62]) Consider two transfer function matrices  $F, G \in RH_\infty$  where  $F$  and  $G$  have the same number of rows, then  $F$  and  $G$  are left-coprime over  $RH_\infty$  iff there exist  $X, Y \in RH_\infty$  such that

$$FX + GY = I \quad (46)$$

These two identities are also called the Bezout or Diophantine equations. Note that state space realizations of coprime factors are readily available but are not included here, see [62], [63]. Finally we define the doubly coprime factorization.

**Lemma 3.2** ([62]) For each proper real-rational matrix  $G$  there exist  $M, \bar{M}, N, \bar{N}, X, \bar{X}, Y$  and  $\bar{Y}$  (all of these matrices in  $RH_\infty$ ) such that:

$$G = NM^{-1} = \bar{M}^{-1}\bar{N} \quad (47)$$

and

$$\begin{pmatrix} \bar{X} & -\bar{Y} \\ -\bar{N} & \bar{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (48)$$

With these factorizations defined, their implications can now be addressed. Let  $G$  be a proper real rational transfer function, then there are factorizations  $G = NM^{-1}$  and  $G = \bar{M}^{-1}\bar{N}$  where  $N$  and  $M$  are right coprime (in  $RH_\infty$ ) and  $\bar{N}$  and  $\bar{M}$  are left coprime (in  $RH_\infty$ ). A doubly coprime factorization finally implies right coprimeness of  $N$  and  $M$  and left coprimeness of  $\bar{N}$  and  $\bar{M}$ . Let right and left coprime factorizations be abbreviated by rcf and lcf respectively.

The following theorem establishes the connection between the described factorizations with the original  $H_\infty$ -problem.

**Theorem 3.2 ([62])** *Let  $K_0$  be a stabilizing controller and define*

$$P_{22} = NM^{-1} = \bar{M}^{-1}\bar{N} \quad (49)$$

$$K_0 = U_0V_0^{-1} = \bar{V}_0^{-1}\bar{U}_0 \quad (50)$$

where  $M, \bar{M}, N, \bar{N}, U_0, \bar{U}_0, V_0$  and  $\bar{V}_0$  represent lcf and rcf of  $P_{22}$  and  $K_0$  then the following statements are equivalent:

1.  $K$  stabilizes  $P$
2.  $\begin{pmatrix} M & U_0 \\ N & V_0 \end{pmatrix}^{-1} \in RH_\infty$ .
3.  $\begin{pmatrix} \bar{V}_0 & -\bar{U}_0 \\ -\bar{N} & \bar{M} \end{pmatrix}^{-1} \in RH_\infty$

Furthermore, the set of all proper stabilizing controllers  $K$  is given by:

$$K = K_0 + \bar{V}_0^{-1}Q(I + V_0^{-1}NQ)^{-1}V_0^{-1} \quad (51)$$

$$= (U_0 + MQ)(V_0 + NQ)^{-1} \quad (52)$$

with  $Q \in RH_\infty$  such that  $(I + V_0^{-1}NQ)(\infty)$  is invertible ( $(\cdot)(\infty)$  denotes the high-frequency gain, i.e. value of the transfer function as  $s \rightarrow \infty$ ).

It is easy to show now that the closed-loop transfer function can be represented as

$$\begin{aligned} \mathcal{F}(P, K) &= P_{11} + P_{12}(I - P_{22}K)^{-1}P_{21} \\ &= \mathcal{F}(T, Q) \\ &= T_{11} + T_{12}QT_{21} \end{aligned} \quad (53)$$

This is the key result of Youla ( $Q$ -) parametrization. Now the original problem is to find a  $Q$  such that

$$\|T_{11} + T_{12}QT_{21}\|_\infty < \gamma, \quad Q \in RH_\infty \quad (54)$$

**Remarks:**

1. Note that  $Q$  is - in general - a transfer function that is restricted to be in  $RH_\infty$ . That is, it is required to be proper and asymptotically stable. Note also that  $\mathcal{F}(T, Q)$  is affine in  $Q$ .

2.  $(I + V_0^{-1}VQ)(\infty)$  invertible is equivalent to having  $(I + D_{22}Q(\infty))$  invertible. This condition is termed 'well posedness' (see e.g. [61]). It reflects the necessity that for non strictly proper controllers the problem is well posed  $\forall s$ .
3. The factorization results used in this presentation are useful as they nicely reflect an 'if and only if' condition for the stabilizability of the system. That is, stability of  $(I - P_{22}K)^{-1}$  can be expressed in terms of the stability of the matrix
 
$$\begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1}$$
 which in turn can be expressed in terms of the above factorizations (for a nice derivation of this connection, see section 6.4.2 of [63]).
4. A similar theorem can be established using coprime factorizations of  $P_{22}$  and  $K_0$ .
5. If  $\mathcal{F}(T, Q)$  is minimized over  $Q$  then this problem is equivalent to the model matching problem (match  $T_{12}QT_{21}$  as close to the model  $T_{11}$  as possible).
6. Under various assumptions on  $T_{ij}$  this problem represents the so called 1-block, 2-block or 4-block problem.
7. Under certain assumptions on  $T_{ij}$  the problem can be transformed into the Hankel Approximation Problem (also called Nehari Extension Problem).
8. As  $Q$  is restricted to be stable and proper, this represents an infinite dimensional optimization problem. Boyd ([1]) used the above parametrization in  $Q$  with a prespecified order (large!) to approximate this problem.
9. For a nice introduction into  $H_\infty$  see [22], [63] or [57] and for a thorough mathematical treatment of this problem, see e.g. [61].

### 3.4 State-Space Formulae for the $H_\infty$ -Bound Problem

Instead of dwelling into the mathematics involved in the solution of the most general case (i.e. the 4-block problem), we will summarize key results that give precise state-space formulae for which the  $H_\infty$ -bound problem can be solved. One of the important results is that all strictly proper stabilizing controllers have the state-space realization of an observer. This fact can be shown using state-space realization of the coprime factors defined above and assembling a complete realization of the controller  $K(s)$ . This structural knowledge has been exploited by Doyle, Glover, Khargonekar, Francis yielding the well-known DGKF-method (see [13],[63]). At this point two different approaches have been considered; one relies on two algebraic Riccati equations and a coupling condition (as pursued by Doyle, Glover, Khargonekar, Francis) which form the basis for most solution of the  $H_\infty$ -bound problem. A summary of this approach can be found in the 1988 paper of Glover and Doyle [16]. The proof to this approach relies heavily on factorization methods, but it also has a nice interpretation in terms of the Youla parametrization and game theoretical results. The other approach taken by Khargonekar and Zhou [21] as well as Sampei and Nakamichi [46] is based on two Riccati inequalities.

### 3.4.1 The DGKF Results

Let us first consider the standard assumptions and state the problem formulation of the DGKF paper [13]. The system under consideration has the state-space realization given for the system  $\Sigma$  (see (42)) with  $w(t) \in R^{m_1}$ ,  $u(t) \in R^{m_2}$ ,  $z(t) \in R^{p_1}$ ,  $y(t) \in R^{p_2}$ ,  $x(t) \in R^n$ . With the following additional assumptions (often termed the standard  $H_\infty$ -assumptions):

- **A1.**  $(A, B_2)$  controllable and  $(C_2, A)$  stabilizable

- **A2.**  $\text{rank}(D_{12}) = m_2$ ,  $\text{rank}(D_{21}) = p_2$

- **A3.**  $D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$ ,

$$D_{21} = \begin{pmatrix} 0 & I \end{pmatrix},$$

$$D_{11} = \begin{pmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{pmatrix}$$

where  $D_{1122}$  has  $m_2$  rows and  $p_2$  columns

- **A4.**  $D_{22} = 0$

- **A5.**  $\text{rank} \begin{pmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m_2 \quad \forall \omega \in \mathcal{R}$

- **A6.**  $\text{rank} \begin{pmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{pmatrix} = n + p_2 \quad \forall \omega \in \mathcal{R}$

#### Remarks:

- Assumption A1 is required to stabilize a possibly unstable plant  $P$ ; equivalently this is necessary for the existence of a stabilizing controller.
- Assumption A2 is a sufficient condition for the existence of a proper controller.
- Assumption A3 can always be achieved by a preliminary scaling and a unitary transformation as long as A2 is satisfied.
- Assumption A4  $D_{22} \neq 0$  can be reincorporated into the controller after that controller is designed for  $D_{22} = 0$ . An alternative route to this 'reincorporation' is given by yet another preliminary unitary transformation as suggested by Stoorvogel [66] that results in a system with  $D_{22} = 0$ .

- Assumptions A5 and A6 require that the transfer functions from  $u$  to  $z$  and from  $w$  to  $y$  have no invariant zeros on the  $j\omega$ -axis. One interpretation of these conditions can be traced back to the Youla parametrization (see e.g. [17]):  $T_{21}$  ( $T_{12}$  respectively) is right (left respectively) invertible in  $RL_\infty$  iff A5 (A6 respectively) is satisfied. This means that this approach can be converted into a Nehari extension problem. Another (state-space) interpretation for these requirements is given by the fact that the two ARE's to be defined have a solution if the associated Hamiltonian has no  $j\omega$ -eigenvalues which in turn is given only if A5 and A6 are satisfied.

With the above assumptions, the existence theorem can be clearly stated.

Let's define

$$D_{1*} = \begin{pmatrix} D_{11} & D_{12} \end{pmatrix}, \quad D_{*1} = \begin{pmatrix} D_{11} \\ D_{21} \end{pmatrix}$$

$$\bar{R} = D_{1*}^T D_{1*} - \begin{pmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{R} = D_{*1} D_{*1}^T - \begin{pmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Finally let  $\rho(\cdot)$  denote the spectral radius (maximum eigenvalue) of its argument, then

**Theorem 3.3** ([13], [16], [63]) *Consider a system  $\Sigma$  as defined above with the assumptions A1 - A6 satisfied, then there exists an internal stabilizing controller  $K(s)$  such that  $\|\mathcal{F}(P, K)\|_\infty < \gamma$  if and only if*

1.  $\gamma > \max\{\bar{\sigma}[D_{1111}, D_{1112}], \bar{\sigma}[D_{1111}^T, D_{1121}^T]\}$
2. there is a real solution  $X_\infty = X_\infty^T \geq 0$  to the ARE

$$X_\infty(A - B\bar{R}^{-1}D_{1*}^T C_1) + (A - B\bar{R}^{-1}D_{1*}^T C_1)^T X_\infty - X_\infty B\bar{R}^{-1}B^T X_\infty + C_1^T(I - D_{1*}\bar{R}^{-1}D_{1*}^T)C_1 = 0 \quad (55)$$

3. there is a real solution  $Y_\infty = Y_\infty^T \geq 0$  to the ARE

$$Y_\infty(A - B_1 D_{*1}^T \hat{R}^{-1} C)^T + (A - B_1 D_{*1}^T \hat{R}^{-1} C)Y_\infty - Y_\infty C^T \hat{R}^{-1} C Y_\infty + B_1(I - D_{*1}^T \hat{R}^{-1} D_{*1})B_1^T = 0 \quad (56)$$

4.  $\rho(X_\infty Y_\infty) < \gamma^2$ .

A state-space realization for the controller is given in the references cited with this theorem. Let us turn to some interpretations:

1. The ARE (55) is frequently called the Generalized Control Algebraic Riccati Equation (GCARE) as  $X_\infty$  is necessary for the existence of a state feedback controller and thus the existence of the feedback gain matrix from the controller to the plant. The ARE (56) on the other hand is frequently called the Generalized Filter Algebraic Riccati Equation (GFARE) as  $Y_\infty$  is necessary for the existence of an observer-based controller in the case of output feedback which then involves an estimation (Luenberger observer) to reconstruct the states that are not directly available from  $y$ . Condition 4 finally reflects the  $H_\infty$ -bound (see [17]).

2. If A3 is not satisfied, this scheme breaks down; this problem is called the singular  $H_\infty$  problem which has been tackled by Stoorvogel in his dissertation ([66]).
3. The derived controller is not unique despite the fact that  $X_\infty$  and  $Y_\infty$  are unique. The 'standard' realization of the controller given in literature is one of many if such a controller exists. If, however, one of the above conditions is violated then there is no stabilizing controller that achieves the  $H_\infty$ -bound. This is the most significant result of this formulation.
4. Also, if a controller is designed using this method for a prespecified  $\gamma$ , nothing can be said about what the actual  $H_\infty$ -norm of the closed loop system is, only that the bound is satisfied.
5. The proof relies heavily on frequency domain factorization ideas, a complete proof can be found in [17].
6. The design procedure depends on whether only a  $H_\infty$ -bound solution is desired or if the  $H_\infty$ -norm of the closed loop system is actually to be minimized. In the first case, only the stated conditions have to be checked for a given  $\gamma$ . In the latter case we have to iterate on  $\gamma$  with decreasing values of  $\gamma$  until one of the above conditions is violated.
7. The controller order is that of the plant so that possible frequency weighting will increase the order of the controller (i.e. plant order plus the sum of the order of all the frequency weights!).

### 3.4.2 The Zhou and Khargonekar Results

As mentioned earlier, the authors examined the  $H_\infty$ -bound problem using the Quadratic Matrix Inequality (17). Their approach is appealing since it does not impose as many restrictions as the DGKF algorithm. Also, the proof uses mostly algebraic methods and is easier to follow. However, the design procedure involves another tuning parameter in addition to  $\gamma$ . The presentation here mainly follows the paper by Sampai, Mita and Nakamichi [46]. Their results in [46] are an extension of the results in [21] where the state feedback case is treated. Let us look at the assumptions made in this approach.

The system under consideration is the state-space realization of the system  $\Sigma$  (see (42)) with  $w(t) \in R^{m_1}$ ,  $u(t) \in R^{m_2}$ ,  $z(t) \in R^{p_1}$ ,  $y(t) \in R^{p_2}$ ,  $x(t) \in R^n$  and the additional assumptions:

**A1**  $(A, B_2)$  controllable and  $(C_2, A)$  observable

**A2**  $D_{22} = 0$

These are the only restrictions on the system model. A1 has already been justified and A2 has already been identified as not being restrictive at all. Clearly this approach can accommodate a far wider class of problems where only observability and controllability of the above stated matrices have to be satisfied; these are standard assumption in any traditional LQ-design. The approach is split into two categories: development of necessary and sufficient conditions



for the strictly proper controller case and its extension to the proper case.

Some results are summarized below. For the state feedback case Khargonekar and Zhou [21] showed when  $C_2 = I$ ,  $D_{21} = 0$  and  $D_{22} = 0$  - the state-feedback configuration - the optimal state feedback gain matrix  $F$  can be characterized in terms of a certain ARE (eqn. 3.3 in [21]). Theorem 3.5 in this reference finally presents a state-space realization of the state-feedback gain matrix  $F$  that achieves the desired  $H_\infty$ -bound. With the above results, the main theorem of [46] can be stated as follows, where  $G_c$  denotes the closed-loop system transfer function matrix between  $w$  and  $z$ .

**Theorem 3.4 ([46], Theorem 1, Corollary 1)** Denote  $\hat{R} = (\gamma^2 - D_{11}^T D_{11})$  and  $\hat{S} = (\gamma^2 - D_{11} D_{11}^T)$ . There is a strictly proper controller which stabilizes the system under consideration and satisfies  $\|G_c\|_\infty < \gamma$  if and only if the following conditions are satisfied:

1.  $\hat{R} > 0$
2. There exists a  $P = P^T > 0$  and a  $Q = Q^T > 0$  satisfying the following conditions:
  - a) There exists a matrix  $F$  such that

$$P[A + B_2 F + B_1 \hat{R}^{-1} D_{11}^T (C_1 + D_{12} F)] + [A + B_2 F + B_1 \hat{R}^{-1} D_{11}^T (C_1 + D_{12} F)]^T P + P B_1 \hat{R}^{-1} B_1^T P + (C_1 + D_{12} F)^T (I + D_{11} \hat{R}^{-1} D_{11}^T) (C_1 + D_{12} F)^T < 0 \quad (57)$$

- b) There exists a matrix  $K$  such that

$$Q[A + K C_2 + C_1^T \hat{S}^{-1} D_{11} (B_1 + K D_{21})^T] + [A + K C_2 + C_1^T \hat{S}^{-1} D_{11} (B_1 + K D_{21})^T]^T Q + (B_1 + K D_{21}) (I + D_{11}^T \hat{S}^{-1} D_{11}) (B_1 + K D_{21})^T < 0 \quad (58)$$

- c)  $\rho(QP) < \gamma^2$

If such matrices exist, then a parametrization of a strictly proper controller is given by

$$\begin{aligned} \dot{\zeta}(t) &= \hat{A} \zeta(t) + \hat{B} y(t) \\ u(t) &= \hat{C} \zeta(t) \end{aligned}$$

where the matrices  $\hat{A}$ ,  $\hat{B}$  and  $\hat{C}$  are defined by equations (8)-(12) in Corollary 1 of [46].

The extension to the proper case is performed by finding a matrix  $\hat{D}$ , the high frequency gain of the proper controller, that maintains the  $H_\infty$ -bound property of the strictly proper controller designed for the system. Theorem 2 and Algorithm 1 in [46] give necessary and sufficient conditions for the existence of such a proper controller as well as a parametrization and a computational 'guideline' for the actual computation of the controller. The matrices  $F$  and  $K$  in above theorem can be computed using Zhou and Khargonekar's procedure in [21]. This computation involves the additional tuning parameter mentioned above.

## 4 Current Approaches in Mixed $H_2/H_\infty$ -Control

As mentioned in the introduction, the  $H_\infty$ -framework only provides robust stability. Robust performance can be incorporated by introducing a 'fictitious' performance  $\Delta$ -blocks or frequency weighting. System inputs and outputs that are associated with system uncertainties and other noises entering the system cannot be modeled independently in a pure  $H_\infty$ -framework. If mixed strategies are desired, i.e. a  $H_2$ -criterion for some criterion outputs  $z_2$  with respect to some disturbance inputs  $w_2$  and at the same time a  $H_\infty$ -criterion for some  $z_\infty$  with respect to some  $w_\infty$ , then we arrive at the problem of mixed  $H_2/H_\infty$ -control. The structural setup for this problem has to be extended to include the new inputs and outputs. The most general description of this problem can be put in the following form:

$$\Sigma_{2/\infty} : \begin{cases} \dot{x}(t) &= \check{A}x(t) + \check{B}_1w_2(t) + \check{B}_2w_\infty(t) + \check{B}_3u(t) \\ z_2(t) &= \check{C}_1x(t) + \check{D}_{11}w_2(t) + \check{D}_{12}w_\infty(t) + \check{D}_{13}u(t) \\ z_\infty(t) &= \check{C}_2x(t) + \check{D}_{21}w_2(t) + \check{D}_{22}w_\infty(t) + \check{D}_{23}u(t) \\ y(t) &= \check{C}_3x(t) + \check{D}_{31}w_2(t) + \check{D}_{32}w_\infty(t) + \check{D}_{33}u(t) \end{cases} \quad (59)$$

Note, that this system description is very general, it can include external disturbances entering the system (modeled by  $w_2$ ), disturbances due to system uncertainties (modeled by  $w_\infty$ ), an output vector  $z_\infty$  representing the output for the feedback closure via the uncertainty system  $\Delta(s)$  and an output vector  $z_2$  representing a set of signals to which the  $H_2$ -objective is applied. It is obvious that this kind of system assumptions allows a more realistic system model than that assumed for pure  $H_\infty$  problems. A feedback controller  $K(s)$  is given by  $u(s) = K(s)y(s)$ . Design objectives are then expressed in terms of a  $H_\infty$ -criterion from  $w_\infty$  to  $z_\infty$  for robust stability and an  $H_2$ -objective for the transfer function from  $w_2$  to  $z_2$ . These design objectives, however, do not take into account cross-couplings from  $w_2$  to  $z_\infty$  and from  $w_\infty$  to  $z_2$ . Due to these cross-couplings a general mixed  $H_2/H_\infty$ -strategy has yet to be defined.

At this point, different design approaches have to be considered separately. They vary in basic system assumptions and theoretical strategy. However, connections have been made recently and they are stated below. In particular, two important subproblems have been investigated recently, namely the two 'disturbance input/one criterion output' case and the 'one disturbance input/two criterion output' case.

In the following subsections, we adopt the notation of the according reference for ease of comparison. Thus, matrices in these subsections might have a different meaning than those defined in other sections. Furthermore the explicit time dependency of signals is suppressed in most formula. If variables are considered in the frequency-domain, they will be shown explicitly.

### 4.1 The Bernstein and Haddad Approach

Researchers Bernstein and Haddad addressed one of the subproblems stated above ([4], [5], [9]). Note that an early paper [4] addressed a slightly different problem than the later papers,

i.e. the direct feedthrough matrix  $E_\infty$  from  $w$  to  $z_\infty$  was assumed to be zero in [4]. The idea however is the same as in their later papers, which are only an extension to the case of a proper transfer function between  $w$  and  $z_\infty$  (with more complicated Riccati equations). To facilitate this presentation, let us consider the case of a strictly proper transfer function between  $w$  and  $z_\infty$  as assumed in [4]. The system under consideration is

$$\Sigma_{2/\infty}^{\text{BH}} : \begin{cases} \dot{x} &= -Ax + D_1 w + Bu \\ z_2 &= E_1 x + E_2 u \\ z_\infty &= E_{1\infty} x + E_{2\infty} u \\ y &= Cx + D_2 w \end{cases} \quad (60)$$

with a strictly proper controller  $C(s)$  of the same order as the plant and it is defined as

$$C(s) : \begin{cases} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c \end{cases} \quad (61)$$

along with the following assumptions and notation,

- 1)  $(A, B)$  and  $(A, D_1)$  are controllable;
- 2)  $(C, A)$  and  $(E_{1\infty}, A)$  are observable;
- 3)  $E_1^T E_2 = 0$ ;  $E_1^T E_1 = R_1$ ;  $E_2^T E_2 = R_2 \geq 0$ ;
- 4)  $E_{1\infty}^T E_{2\infty} = 0$ ;  $E_{1\infty}^T E_{1\infty} = R_{1\infty}$ ;  $E_{2\infty}^T E_{2\infty} = R_{2\infty}$ ;
- 5)  $D_1 D_2^T = 0$ ;  $D_1 D_1^T = V_1$ ;  $D_2 D_2^T = V_2 > 0$ ;
- 6)  $\tilde{V} = \begin{pmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{pmatrix}$ ;  $\tilde{R}_\infty = \begin{pmatrix} R_{1\infty} & 0 \\ 0 & C_c^T R_{2\infty} C_c \end{pmatrix}$ ;  $\tilde{R} = \begin{pmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{pmatrix}$

This approach is very appealing as it deals with one disturbance input vector  $w$  plays a dual role depending on which output vector and, thus, which design objective is considered.

Let the closed-loop system - with the controller in place - be written in the following form,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{D}w \quad (62)$$

$$z_2 = \tilde{E}\tilde{x} \quad (63)$$

$$z_\infty = \tilde{E}_\infty \tilde{x} \quad (64)$$

where  $\tilde{x}$  contains the plant and controller states, the above state model matrices can easily be computed from (60) and (61). Let furthermore the closed-loop transfer function from  $w$  to  $z_\infty$  be denoted by  $H(s)$ ,

$$H(s) = \tilde{E}_\infty (sI - \tilde{A})^{-1} \tilde{D} \quad (65)$$

then the design objectives can be stated as follows,

- i) Find a strictly proper controller such that the closed-loop system from  $w$  to  $z_\infty$  is asymptotically stable, and

ii) the closed-loop transfer function between  $w$  and  $z_\infty$  satisfies

$$\|H(s)\|_\infty < \gamma \quad (66)$$

where  $\gamma$  is a prespecified constant, and

iii) the controller minimizes the cost function

$$\begin{aligned} J(A_c, B_c, C_c) &= \lim_{t \rightarrow \infty} \mathcal{E}[\tilde{x}^T \tilde{E}^T \tilde{E} \tilde{x}] \\ &= \lim_{t \rightarrow \infty} \mathcal{E}[\tilde{x}^T \tilde{R} \tilde{x}] \end{aligned} \quad (67)$$

**Remarks:**

- For the design objective ii) the disturbance is assumed to be  $w \in L_2$ .
- In iii) it is assumed that  $w$  is white gaussian noise with unit spectral density.
- In iii)  $\mathcal{E}$  denotes the expectation operator.

The solution approach relies on the formulation of an auxiliary problem for the  $H_2$ -problem that 'automatically' enforces the  $H_\infty$ -constraint in (66). To do that let us examine the cost function  $J(A_c, B_c, C_c)$  in more details. It represents simply a weighted steady-state state covariance responses of the closed-loop system to the disturbances  $w$  being white noises and having a unit power spectral density matrix. It follows from a standard result that

$$J(A_c, B_c, C_c) = \text{Trace}\{\tilde{Q}\tilde{R}\} \quad (68)$$

where  $\tilde{Q}$  is the positive definite solution of the Lyapunov equation

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0 \quad (69)$$

where  $\tilde{A}$  is asymptotically stable. Now we define the following Riccati equation

$$\tilde{A}Q + Q\tilde{A}^T + \gamma^{-2}Q\tilde{R}_\infty Q + \tilde{V} = 0 \quad (70)$$

It can be verified that, if a solution  $Q$  for the ARE in (70) exists - assuming the associated  $\tilde{A}$  is asymptotically stable and the above controllability assumption -, then this implies  $\|H(s)\|_\infty < \gamma$  and  $Q - \tilde{Q} \geq 0$  (Lemma 2.1 in [4]). These results motivate the definition of a new cost function that incorporates the  $H_\infty$ -bound into the  $H_2$ -optimization. This new cost function defined by Bernstein and Haddad is defined in terms of  $Q$  instead of  $\tilde{Q}$ . Namely,

$$\mathcal{J}(A_c, B_c, C_c, Q) = \text{Trace}\{Q\tilde{R}\} \quad (71)$$

It can be easily verified that

$$\mathcal{J}(A_c, B_c, C_c, Q) \geq J(A_c, B_c, C_c) \quad (72)$$

From  $Q - \tilde{Q} \geq 0$  it is obvious that the new cost function represents an upper limit for the 2-norm of the closed-loop system so that the whole setup considered by Bernstein and

Haddad is suboptimal with respect to the 2-norm minimization as well as in the  $H_\infty$ -sense (since only a bound for the norm  $\|H(s)\|_\infty$  is considered).

The new auxiliary problem can now be described as "Minimize the new cost function  $\mathcal{J}(A_c, B_c, C_c, Q)$  subject to (70)".

The further development is based on forming the Lagrangian function for the auxiliary optimization problem by appending the ARE (70) to the cost function  $\mathcal{J}(A_c, B_c, C_c, Q)$ . This yields

$$\mathcal{L}(A_c, B_c, C_c, Q, \mathcal{M}) = \text{Trace}\{Q\ddot{R} + [\ddot{A}Q + Q\ddot{A} + \gamma^2 Q\ddot{R}_\infty Q + \ddot{V}]\mathcal{M}\} \quad (73)$$

Derivation of this Lagrangian with respect to  $Q, A_c, B_c, C_c$  and  $\mathcal{M}$  yields the final result. The lemma stated below is taken from [56] and its implications will be discussed later.

**Lemma 4.1** ([56], Lemma 1) *Let's consider a plant given by (61) and the according assumptions stated above. Let  $\gamma$  be a prespecified positive constant. Furthermore, let  $C(s)$  be a strictly proper stabilizing controller of the same order as the plant that minimizes the cost function  $\mathcal{J}(A_c, B_c, C_c, Q)$  subject to  $\|H(s)\|_\infty < \gamma$ . Then such a controller exists if and only if there exist real symmetric matrices  $Q \geq 0$ ,  $P \geq 0$  and  $\hat{Q} \geq 0$  satisfying:*

$$0 = AQ + QA^T + Q[\gamma^{-2}R_{1\infty} - C^T V_2^{-1} C]Q \quad (74)$$

$$0 = [A + \gamma^{-2}(Q - \hat{Q})R_{1\infty}]^T P + P[A + \gamma^{-2}(Q - \hat{Q})R_{1\infty}] + E_1^T E_1 - C_c^T R_2 C_c \quad (75)$$

$$0 = [A + \gamma^{-2}QR_{1\infty} + BC_c]\hat{Q} + \hat{Q}[A + \gamma^{-2}QR_{1\infty} + BC_c]^T + \gamma^{-2}\hat{Q}[R_{1\infty} + C_c^T R_{2\infty} C_c]\hat{Q} + QC^T V_2^{-1} CQ \quad (76)$$

$$A_c = A - B_c C + BC_c + \gamma^{-2}QR_{1\infty} \quad (77)$$

$$B_c = QC^T V_2^{-1} \quad (78)$$

$$0 = R_2 C_c + B^T P + \gamma^{-2}R_{2\infty} C_c \hat{Q} P \quad (79)$$

$$Q = \begin{pmatrix} Q + \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{pmatrix} \quad (80)$$

#### Remarks:

1. Note that in the original paper of Bernstein and Haddad ([4]) these conditions were labeled 'necessary'. In a very recent paper by Yeh, Banda and Chang ([56]), it has been shown that these conditions are also sufficient (see Corollary 1.1 and Theorem 2 in [56]). This justifies the 'if and only if' relation in above lemma.
2. The above theory has also been extended to the low-order strictly proper controller case which adds yet another Lyapunov-type function to the above conditions. This additional function is also coupled with the other equations (see Theorem 6.1 in [4]).
3. Presently only numerical methods are available to solve the system of equations of Lemma 4.1 (see e.g. [10] or [11]) - under the assumption that such a solution exists.

## 4.2 The Doyle, Zhou and Bodenheimer Approach

Researchers Doyle, Zhou and Bodenheimer have considered the dual problem to that of Bernstein and Haddad ([14], [15]). The problem is dual with respect to the system assumptions as well as the necessary and sufficient conditions derived for their problem. The system under consideration is

$$\Sigma_{2/\infty}^{DZB} : \begin{cases} \dot{x} &= Ax + B_0 w_0 + B_1 w_1 + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{20} w_0 + D_{21} w_1 \end{cases} \quad (81)$$

with a strictly proper full-order controller  $K(s)$

$$K(s) : \begin{cases} \dot{x}_c &= A_k x_c - L y \\ u &= F_\infty x_c \end{cases} \quad (82)$$

### Analysis of a System with Two Disturbance Inputs

In this presentation, only an outline of the approach and some of the important conclusions are described. Necessary and sufficient conditions can be derived using Bernstein and Haddad's results. Let  $G_0$  denote the transfer function from  $w_0$  to  $z$  and  $G_1$  the transfer function from  $w_1$  to  $z$ . The cost function is now defined by the output power  $\|z\|_P^2$ :

$$J = \sup_{w_0 \in BS, w_1 \in BP} \|z\|_P^2 \quad (83)$$

Note that the white noise input  $w_0$  constitutes the worst-case disturbance for signals with bounded spectrum, despite the fact that it is not in  $BP$  and hence not in  $S$  as it does not have bounded power. However, the authors of this paper assume that the results derived exist in the limit for signals in  $BS \rightarrow$  white noise. Formally they do treat white noise as a set member of  $BS$  and assume white noise with unity spectral density  $S_{w_0 w_0} = I$  as the worst-case disturbance for  $w_0$ .

Let us at this point assume that a certain controller is in place so that the closed-loop system can be written as

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_0 w_0 + \hat{B}_1 w_1 \\ z &= \hat{C}\hat{x} + \hat{D}_1 w_1 \end{aligned} \quad (84)$$

where  $\hat{x}$  contains  $x$  and  $x_c$ . To illustrate the schematics, we make the assumption  $\hat{D}_1 = 0$  (Note that the proper case is treated in [15] and involves a more complicated set of Riccati equations). Then the closed-loop transfer function from  $w_1$  to  $z$  is simply given by  $G_1(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}_1$ . Under this additional assumption, the following theorem can be verified.

**Theorem 4.1** ([14], Theorem 1 and Theorem 2) *Suppose  $\gamma > \|G_1\|_\infty$ , then*

$$\sup_{w_0 \in BS} [\|z\|_P^2 - \gamma^2 \|w_1\|_P^2] = \text{Trace}[\hat{B}_0^T X_\gamma \hat{B}_0] - \|w_1 - \gamma^{-2} \hat{B}_1^T X_\gamma \hat{x}\|_P^2 \quad (85)$$

$$\sup_{w_1 \in BP, w_0 \in BS} [\|z\|_P^2 - \gamma^2 \|w_1\|_P^2] = \text{Trace}[\hat{B}_0^T X_\gamma \hat{B}_0] \quad (86)$$

with  $w_{1,worst} = \gamma^{-2} \hat{B}_1^T X_\gamma \hat{x}$  and  $X_\gamma$  is the real symmetric positive definite solution of

$$\hat{A}^T X_\gamma + X_\gamma \hat{A} + \gamma^{-2} X_\gamma \hat{B}_1 \hat{B}_1^T X_\gamma + \hat{C}^T \hat{C} = 0 \quad (87)$$

such that  $(\hat{A} - \gamma^{-2} \hat{B}_1^T X_\gamma)$  is asymptotically stable and the parameter  $\gamma$  has to be chosen large enough so that  $\|w_1\|_P^2 = \|\gamma^{-2} \hat{B}_1^T X_\gamma \hat{x}\|_P^2 \leq 1$ .

Note that the last assumption assures  $w_1 \in B\mathcal{P}$ . Furthermore the factorization  $w_1 = K\hat{x}$  can be motivated again. The actual cost  $J$  as defined in (83) can be computed using the observability grammian or the controllability grammian of the closed-loop system with  $w_{1,worst}$  in place.

### Synthesis in a System with Two Disturbance Inputs

With these considerations the actual problem statement - the synthesis problem - as approached by Doyle, Zhou and Bodenheimer can be stated as

Given a plant  $\Sigma_{2/\infty}^{DZB}$  and a prespecified  $\gamma$ , find an internally stabilizing controller  $K(s)$  that solves the mixed  $H_2/H_\infty$ -control problem:

$$\min_{K(s)} \sup_{w_1 \in \mathcal{P}, w_0 \in BS} [\|z\|_P^2 - \gamma^2 \|w_1\|_P^2] \quad (88)$$

It is obvious that if we restrict  $\|w_1\|_P^2 = 1$ , then this cost function actually solves the problem

$$J_{DZB} = \inf_{K(s)} \sup_{w_0 \in BS, w_1 \in B\mathcal{P}} \|z\|_P^2 \quad (89)$$

Note that the supremum in the above problem statement is defined over  $w_1 \in \mathcal{P}$  and not over  $w_1 \in B\mathcal{P}$  as this would involve an ' $\gamma$ -iteration' to force  $w_1$  into  $B\mathcal{P}$ . As long as the closed-loop system is stable,  $w_1$  will always be in  $\mathcal{P}$ . The optimal mixed  $H_2/H_\infty$ -controller is defined for a  $\gamma$  that yields  $w_1 \in B\mathcal{P}$ . It has to be kept in mind that the truly optimal mixed  $H_2/H_\infty$ -synthesis problem is defined for  $\|w_1\|_P^2 = 1$ . It is clear from the cost function  $J_{DZB}$  that, for arbitrary  $\gamma$ ,  $J_{DZB}$  reduces to the pure  $H_2$ -norm minimization problem if  $w_1 = 0$ , and it reduces to the standard  $H_\infty$ -norm bound problem if  $w_0 = 0$ .

The actual synthesis of the controller is performed by separating the problem into a so-called Mixed Full Control problem and a Mixed Output Estimation problem, corresponding respectively to the computation of an optimal feedback gain matrix and the design of an optimal estimator. This procedure will not be covered here as the resulting equations are dual to the equations derived by Bernstein and Haddad.

In the paper [56] by Yeh, Banda and Chang, it is proved that the two approaches of Bernstein/Haddad and Doyle, Zhou and Bodenheimer yield identical necessary and sufficient conditions if the Bernstein-Haddad approach is applied to the dual system of  $\Sigma_{2/\infty}^{DZB}$ . That is, perform the following change of notation in the above Bernstein-Haddad approach:

$$A \rightarrow A^T; \quad E_1 \rightarrow B_0^T; \quad C \rightarrow B_2^T;$$

$$D_1 \rightarrow C_1^T; \quad D_2 \rightarrow D_{12}^T;$$

$$\begin{aligned}
B &\rightarrow C_2^T; & E_2 &\rightarrow D_{20}^T; & E_{2\infty} &\rightarrow D_{2\infty}^T; \\
A_c &\rightarrow A_k^T; & C &\rightarrow L^T; & B_c &\rightarrow -F_\infty^T; \\
R_2 &\rightarrow R_0; & R_{2\infty} &\rightarrow R_2; & V_2 &= I \text{ (pure scaling)}
\end{aligned}$$

and

$$Q \rightarrow X_\infty; \quad \hat{Q} \rightarrow Y; \quad Q \rightarrow X_\gamma$$

Then the equations derived from this new dual system by applying Bernstein and Haddad's approach yield exactly the same necessary and sufficient conditions derived by Doyle, Zhou and Bodenheimer in [14] so that the actual controller computation is reduced to the same coupled equations as derived in [4]. This is a very interesting result. The importance of the work of Doyle, Zhou and Bodenheimer in this field is in the interpretation and analysis of the type of disturbance signals in mixed  $H_2/H_\infty$ -analysis.

Finally an observation made by Doyle, Zhou and Bodenheimer is that the mixed  $H_2/H_\infty$ -control problem is solvable only if the pure  $H_\infty$ -control with the chosen  $\gamma$  is solvable (see [15], Lemma 1). This is an important and intuitive result. If there is no controller that solves the suboptimal  $H_\infty$ -bound problem for the chosen  $\gamma$  then there will be no mixed  $H_2/H_\infty$ -controller achieving the mixed  $H_2/H_\infty$ -strategy.

### 4.3 The Rotea and Khargonekar Results

#### 4.3.1 A Bernstein-Haddad Equivalent Setup with State and Static Output Feedback

Researchers Rotea and Khargonekar ([18], [25], [26], [27]) addressed two subproblems of the above general mixed  $H_2/H_\infty$ -problem. The first problem is similar to that of Bernstein and Haddad, namely

$$\Sigma_{2/\infty}^{\text{RK1}} : \begin{cases} \dot{x} &= Ax + B_1 w + B_2 u \\ z_2 &= C_0 x + D_0 u \\ z_\infty &= C_1 x + D_1 u \\ y &= C_2 x + D_2 w \end{cases} \quad (90)$$

Their results are important because they provided a controller factorization that results in a convex optimization problem for the cases of state feedback or static output feedback. The general problem statement that Rotea and Khargonekar addressed can be described as follows,

"For a plant  $\Sigma_{2/\infty}^{\text{RK1}}$  find an output feedback controller  $u(s) = K(s)y(s)$  such that  $\|T_{z_2 w}\|_2 < \alpha$  subject to internal stability and  $\|T_{z_\infty w}\|_\infty < \gamma$ ."

Note that this problem has a slightly different design goal since it tries to satisfy a 2-norm bound instead of a minimization of  $\|T_{z_2 w}\|_2$ . However, the mathematics involved are almost



identical to Bernstein and Haddad's approach except for the parametrization of the controller  $K$ . Note that here the case of  $\gamma = 1$  is considered which obviously can always be achieved by appropriate scaling.

Before we examine the more general case of output feedback, let us look at the simpler problem of state feedback. That is  $C_2 = I$ ,  $D_2 = 0$  and a state-feedback controller  $u(s) = K(s)y(s)$ . Then the closed-loop system is defined by

$$\dot{x} = (A + B_2 K)x + B_1 w \quad (91)$$

$$z_2 = (C_0 + D_0 K)x \quad (92)$$

$$z_\infty = (C_1 + D_1 K)x \quad (93)$$

Now, following Bernstein and Haddad, the condition  $\|T_{z_\infty w}\|_\infty < 1$  is ensured if a real symmetric positive definite solution  $Y$  exists for the following Riccati equation

$$[A + B_2 K]Y + Y[A + B_2 K]^T + Y[C_1 + D_1 K]^T[C_1 + D_1 K]Y + B_1 B_1^T < 0 \quad (94)$$

and  $(A + B_2 K)$  asymptotically stable. Define an upper bound for the 2-norm from  $w$  to  $z_2$  with the cost function

$$J(K) = \text{Trace}[(C_0 + D_0 K)Y(C_0 + D_0 K)^T] \geq \|T_{z_2 w}\|_2 = \text{Trace}[(C_0 + D_0 K)L_c(C_0 + D_0 K)^T] \quad (95)$$

So far the results are similar to those of Bernstein and Haddad for the given system. The significant difference is in the controller parametrization. Let

$$K = WY^{-1} \quad (96)$$

where  $W$  is a set of matrices of appropriate dimensions. Note that the matrix  $Y$  is symmetric and of dimension  $n \times n$ . Let  $x \in R^n$ , and  $u \in R^q$  and define the following set

$$\Omega := \{(W, Y) \in R^{q \times n} \times \Theta : Y > 0\}$$

where  $\Theta$  is the set of  $n \times n$  symmetric matrices. If we put this controller into the cost function (95) and the corresponding Riccati equation (94), then the following functions can be defined

$$f(W, Y) = \text{Trace}[(C_0 + D_0 WY^{-1})Y(C_0 + D_0 WY^{-1})^T] \quad (97)$$

$$Q(W, Y) = AY + YA^T + B_2 W + W^T B_2^T + B_1 B_1^T + (C_1 Y + D_1 W)^T (C_1 Y + D_1 W) \quad (98)$$

We can now define the new optimization problem as

"Find  $K$  such that  $\inf f(W, Y) < \alpha$  subject to  $Q(W, Y) < 0$  with  $(W, Y) \in \Omega$ ."

Next we cite some properties of this optimization problem which have been proven in [27]:

1.  $f(W, Y)$  is a real analytic convex function on  $\Omega$ , this is the major contribution of this factorization.
2.  $Q(W, Y)$  is a convex matrix valued mapping from  $\Omega \rightarrow \Theta$  if  $Y$  is positive definite.
3. The set  $\{(W, Y) \in \Omega : Q(W, Y) < 0\}$  is bounded if  $D_1$  has full column rank and  $(\tilde{C}_1, -\tilde{A})$  is detectable, where  $\tilde{C}_1 = (I - D_1(D_1^T D_1)^{-1} D_1^T)C_1$ , and  $\tilde{A} = A - B_2(D_1^T D_1)^{-1} D_1^T C_1$ .

4. Under these additional conditions the new optimization problem is a finite dimensional convex optimization problem on a bounded domain, a very nice property.

The problem of a static output feedback design in the general system as posed in  $\Sigma_{2/\infty}^{\text{RK1}}$  can be solved by defining an auxiliary system that has a 'state feedback form' as a function of a Riccati equation which in turn depends on the chosen  $\gamma$ . In this manner, the static output feedback case is reduced to the state feedback case.

#### 4.3.2 A System with Two Inputs and Two Outputs

The more general problem of 'simultaneous'  $H_2/H_\infty$ -control ([26]) involving a system with two disturbance inputs and two criterion outputs has been considered. It is given as follows

$$\Sigma_{2/\infty}^{\text{RK2}} : \begin{cases} \dot{x} &= Ax + B_1 w_2 + B_2 w_\infty + B_3 u \\ z_2 &= C_1 x + D_1 u \\ z_\infty &= C_2 x + D_2 u \\ y &= x \end{cases} \quad (99)$$

It can be seen that this approach is also restricted to the state feedback case. Let  $n$  denote the dimension of the above system. Let  $T_2$  denote the transfer function between  $w_2$  and  $z_2$ , and  $T_\infty$  the transfer function between  $w_\infty$  and  $z_\infty$ . At this point it should be mentioned that all the previous attempts to solve the mixed  $H_2/H_\infty$ -problem are set up as constrained optimization problems. That is: "Find a controller that minimizes  $\|T_2\|_2$  and  $\|T_\infty\|_\infty < \gamma$ ". The approach taken in this work is of a different nature.

"The controller to be found is admissible if it minimizes  $\|T_2\|_2$  such that the  $\|T_\infty\|_\infty < 1$ ". The latter formulation represents an unconstrained optimization problem. The approach utilizes a characterization of all dynamic state feedback controllers that minimize  $\|T_2\|_2$ . Among these controllers then the ones that satisfy the additional  $H_\infty$ -constraint are selected. The basis for their analysis is the fact that the controller that minimizes  $\|T_2\|_2$  is not necessarily unique. Let us look at the assumptions in their approach first.

**A1:**  $(A, B_3)$  is stabilizable,

**A2:**  $D_1$  and  $D_2$  have full column rank,

**A3:** The transfer functions from  $u$  to  $z_2$  and from  $u$  to  $z_\infty$  have no invariant zeros on the imaginary axis,

**A4:**  $D_2^T [C_2 \ D_2] = [0 \ I]$ .

These are standard assumptions to ensure that an optimal LQ-state feedback controller exists. Let's  $(.)^+$  denote the Moore Penrose inverse, i.e.  $XX^+X = X$  and  $X^+XX^+ = X^+$  and define the following matrices and Riccati equation

$$0 = A^T X_F + X_F A - (D_1^T C_1 + B_3^T X_F)^T (D_1^T D_1)^{-1} (D_1^T C_1 + B_3^T X_F) + C_1^T C_1 \quad (100)$$

$$F = -(D_1^T D_1)^{-1} (D_1^T C_1 + B_3^T X_F) \quad (101)$$

$$\Pi_1 = I - B_1 B_1^+ \quad (102)$$

$$A_F = A + B_3 F \quad (103)$$

$$C_{1F} = C_1 + D_1 F \quad (104)$$

$$C_{2F} = C_2 + D_2 F \quad (105)$$

where  $X_F$  is the unique solution of ARE (100) and  $F$  is the state feedback matrix that minimizes  $\|T_2\|_2$ . However, this is only one possible solution if  $B_1$  is not of full row rank. The class of all minimizing controllers  $K(s)$  can be characterized in terms of 2 transfer function matrices. For that purpose we define the set  $S$  of transfer functions as

$$S = \{Q \in RH_\infty : Q = W\Pi_1(sI - A_F), W \in RH_2\}. \quad (106)$$

Then Rotea and Khargonekar have shown in Theorem 1 of [25] that the class of all dynamic state feedback controllers that minimize  $\|T_2\|_2$  is given by

$$K^2(s) : \begin{cases} \dot{x}_K &= A_F x_K + B_3 r \\ u &= Fy + Ir \\ v &= -Ix_K + Iy \\ r &= Qv \\ Q &\in S \end{cases} \quad (107)$$

The proof for this theorem consists of first using the Youla parametrization to show that the defined class of controllers is actually stabilizing the plant and then this parametrization is used to show that  $Q \in S$  is the class of controllers that minimizes  $\|T_2\|_2$ . It is easily verified now that if  $\text{im}(B_1) = R^n$  then  $\Pi_1 = 0$  and the above class of minimizing controllers reduces to the minimizing static feedback 'controller'  $F$  ( $\text{im}(\cdot)$  denotes the image space of the argument). In this case, this design method will result in  $F$  as the optimal state feedback matrix. The condition of whether or not  $\|T_\infty\|_\infty < \gamma$  is satisfied, has to be checked via the according Riccati equation or the respective hamiltonian. On the other hand, If  $\text{im}(B_1)$  is a proper subspace of  $R^n$  then the freedom given by the family of controllers in (107) can be used to satisfy the additional  $H_\infty$ -requirement. The following theorem characterizes the class of all dynamic state feedback controllers that satisfy the  $H_\infty$ -requirement in addition.

**Theorem 4.2 (Theorem 2 in [25])** *Given a system  $\Sigma_{2/\infty}^{\text{RK2}}$  as defined above, the problem of minimizing  $\|T_2\|_2$  such that  $\|T_\infty\|_\infty < 1$  is solvable if and only if the following conditions are satisfied:*

1. The ARE

$$A^T X_{2\infty} + X_{2\infty} A + X_{2\infty} (B_2 B_2^T - B_3 B_3^T) X_{2\infty} + C_2^T C_2 = 0 \quad (108)$$

has a stabilizing positive semi-definite solution  $X_{2\infty}$ ,

2. The ARE

$$Y_{2\infty} A_F^T + A_F Y_{2\infty} + Y_{2\infty} C_{2F}^T C_{2F} Y_{2\infty} + B_2 (I - V_2^+ V_2) B_2^T = 0 \quad (109)$$

has a stabilizing solution  $Y_{2\infty}$ .

3.

$$\rho(Y_{2\infty}X_{2\infty}) < 1. \quad (110)$$

where  $V_2 = \Pi_1 B_2$ .

If these conditions are satisfied then the class of all optimal  $H_2/H_\infty$ -optimal dynamic state feedback controllers - optimal in the sense as defined above - is given by

$$\mathbf{K}^{2\infty}(\mathbf{s}) := (A^{2\infty}, B^{2\infty}, C^{2\infty}, D^{2\infty}).$$

where

$$\begin{aligned} A^{2\infty} &= A + (I - \Theta)B_3H + \Theta B_3F + (I - \Theta)B_2B_2^T X_{2\infty} \\ B^{2\infty} &= A^{2\infty}\Theta - \Theta A_F \\ C^{2\infty} &= H - F \\ D^{2\infty} &= F(I - \Theta) + H\Theta \\ \Theta &= Z_2 B_2 V_2^+ \Pi_1 \\ H &= -B_3^T X_{2\infty} \\ Z_2 &= (I - Y_{2\infty}X_{2\infty})^{-1} \end{aligned}$$

#### Remarks:

1. The setup considers the 'nonsingular' case, that is  $D_1^T D_1 \neq 0$ .
2. Solvability of ARE (108) ensures that  $\|T_\infty\|_\infty < 1$  (see [13], Theorem 2).
3. ARE (109) and equation (110) represent conditions that ensure the existence of a controller that solves the problem of minimizing  $\|T_2\|_2$ .
4. Note that  $V_2 = \Pi_1 B_2 = (I - B_1 B_1^+) B_2$ . Thus  $V_2 = 0$  and  $V_2^+ = 0$  if either  $\text{im}(B_1) \in R^n$  or  $\text{im}(B_2) \subset \text{im}(B_1)$ . In both cases, the optimal controller  $\mathbf{K}^{2\infty}(\mathbf{s})$  reduces to the static state feedback controller  $\mathbf{K}^{2\infty}(\mathbf{s}) = F$ .
5. A design algorithm has to solve ARE (100) to find  $F$ , then Theorem 4.2 can be applied to find the optimal mixed  $H_2/H_\infty$ -controller.
6. This approach only considers the case of static or dynamic state feedback.

## 4.4 Other Approaches in Mixed $H_2/H_\infty$ -Control

### 4.4.1 Mustafa's Entropy Function

Before we present our design approach to the mixed  $H_2/H_\infty$ -control problem, the so-called entropy (at infinity, see [64]) of a system needs to be defined.

**Definition 4.1** (Definition 2.2.1 in [64], see also [57]) *Consider a system  $H:=(A,B,C,D)$  with  $H \in RL_\infty$  if  $\|H\|_\infty < \gamma$  then the  $\gamma$ -entropy of this system (at infinity)  $I(H, \gamma)$  is defined as*

$$I(H, \gamma) = -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln |\det\{I - \gamma^{-2} H^*(j\omega)H(j\omega)\}| d\omega \quad (111)$$

If  $\|H\|_\infty \geq \gamma$  then  $I(H, \gamma)$  is unbounded (infinite). Thus, the entropy of the considered system is finite if and only if the chosen  $\gamma$  is larger than  $\|H\|_\infty$ . This fact establishes the connection of entropy to the  $H_\infty$ -framework. Connection to  $H_2$  can be established using Lemma A.2.1 in [64] which states that

$$-\ln |(I - \varepsilon^2 N^* N)| \geq \varepsilon^2 \text{Trace}\{N^* N\}$$

With the above lemma it can be verified that the entropy of a strictly proper system is an upper bound for its 2-norm. That is, (see Theorem 2.4.4 in [64]) let  $H$  be strictly proper and  $\|H\|_\infty < \gamma$ , then  $I(H, \gamma) \geq \|H\|_2^2$ . It can further be shown that  $\lim_{\gamma \rightarrow \infty} I(H, \gamma) = \|H\|_2^2$ . From now on, entropy will denote the entropy at infinity as defined by Mustafa and Glover in [64]. It is obvious that the entropy as defined represents yet another characterization of an  $H_\infty$ -norm bound. The cost function used for mixed  $H_2/H_\infty$ -control finally is (see Definition 1 in [31]) as follows.

**Definition 4.2** ([31]) *Consider a system  $H(s) = [H_0(s) \ H_1(s)]$  with  $H_0(s) \in RH_2$ ,  $H_1(s) \in RH_\infty$  and  $\|H_1\|_\infty < \gamma$ , then we define*

$$L(H, \gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\{(I - \gamma^{-2} H_1 H_1^*)^{-1} H_0 H_0^*\} d\omega \quad (112)$$

In their paper [31], MacMartin, Hall and Mustafa have shown that  $L(H, \gamma)$  also represents an upper bound for the 2-norm of  $H_0$ ; namely,  $L(H, \gamma) \geq \|H_0\|_2^2$  and  $\lim_{\gamma \rightarrow \infty} L(H, \gamma) = \|H_0\|_2^2$ . Thus, for  $\gamma \rightarrow \infty$  the defined cost function represents a pure  $H_2$  cost function. For a finite  $\gamma$ ,  $L(H, \gamma)$  represents a compromise between an  $H_2$ -strategy on  $H_0$  and an  $H_\infty$ -strategy on  $H_1$ . The best  $H_\infty$ -strategy for  $\|H_1\|_\infty < \gamma$  is obviously achieved if all the singular values of  $H_1$  are as small as possible, and thus  $\|H_1\|_\infty$  is as small as possible. The  $H_2$ -strategy is pursued by the fact that  $L(H, \gamma)$  overbounds  $\|H_0\|_2^2$ .

The objective is to minimize this cost function for a certain  $\gamma$  assuming that  $\|H_1\|_\infty < \gamma$ . The actual minimization of  $L(H, \gamma)$  is performed using a state-space approach where the cost is evaluated in terms of these state-space matrices. The system under consideration is identical to  $\Sigma_{2/\infty}^{DZB}$  defined above.  $H_0$  and  $H_1$  represent the transfer functions from  $w_0$  and  $w_1$  to  $z$  respectively. The mathematical optimization is performed similar to the approach taken by Bernstein and Haddad by appending the  $H_\infty$ -ARE to the cost function via a Lagrange multiplier matrix. Derivatives of the resulting Lagrangian with respect to all the respective matrices yield the required necessary conditions consisting of two ARE's and two Lyapunov equations that are coupled. Solutions for these coupled equations are also based on numerical methods.

#### 4.4.2 Steinbuch and Bosgra

Steinbuch and Bosgra addressed both setups (i.e. two disturbance inputs/ one criterion output as well as one disturbance input/two criterion outputs) as defined by Bernstein/Haddad and Doyle/Zhou/Bodenheimer. For both systems they consider the static output feedback case. The first considered system is of the form (S1) with a cost function similar to that defined in Theorem 4.1. The difference to the Doyle/Zhou/Bodenheimer approach is the explicit assumption that  $w_1 = Kx$  where  $x$  is the system state. This assumption is justified by the form of  $w_{1\text{worst}}$  as established in Theorem 4.1. The procedure involves once again the forming of a Lagrangian and the subsequent derivation of necessary conditions. These necessary conditions involve coupled quadratic matrix equations that have to be solved numerically. A solution for the setup of Bernstein and Haddad follows by duality. The extension to fixed order dynamic feedback has also been pointed out.

An interesting comment was made at the end of this paper, namely the authors suggested that the worst case  $H_\infty$ -disturbance can be defined using an observer-based 'disturbance generator'.

#### 4.5 Summary

From the above presentation one can see that all analytical methods currently available are based in one way or another on a set of Riccati (and Lyapunov) equations. Except for the work of Rotea and Khargonekar (the two input/two disturbance case with dynamic state feedback compensators), these resulting equations are highly coupled and can be solved only using numerical methods. Also, all the approaches assume white noise as the worst-case disturbance for the  $H_2$ -optimal strategy. In many cases, it is desirable to find a controller that optimizes the system with respect to different disturbance distributions. This design objective has not yet been incorporated into these approaches.

An important fact can be deduced from these approaches. In the 'two disturbance inputs/one criterion output' case as well as in the 'one disturbance input/two criterion outputs' case, the worst-case disturbance for the  $H_\infty$ -strategy can be expressed as a linear combination of the system state  $x$ . In the more general case of the 'two disturbance inputs/two criterion outputs' Steinbuch and Bosgra suggested the use of a dynamical system to generate the worst-case disturbance.

### 5 On the Cost Functional $\int_0^{t_f} \{z^T(t)z(t) - \gamma^2 w^T(t)w(t)\} dt$

The above cost functional is closely related to equation (20). The relation has been established by Craven (see [60] on pages 108-109). A time-domain definition of the  $H_\infty$ -norm (in equation (20) in terms of a LQ-type function can be done using a Lagrange multiplier  $\nu$  to include the constraint  $\|w\|_2 = 1$  into the cost  $\|z\|_2$ . We have

$$\|H\|_\infty^2 = \sup_{\|w(t)\|_2=1} \lim_{t_f \rightarrow \infty} \int_0^{t_f} \{z^T(t)z(t) + \nu[w^T(t)w(t) - 1]\} dt$$

The constraint  $\|w(t)\|_2 = 1$  is achieved in an iterative manner. If the problem is to determine whether  $\|H\|_\infty < \gamma$ , the function defined in the title of this section is very valuable. Let

$w(t)$  denote the input and  $z(t)$  the output of a linear time invariant system. A more recent analysis of this cost function has been performed by G. Tadmor ([48], [49] and [50]) and a clear connection between inequality (21) and the above cost function has been established. Let in this section  $\|v\|_2^2 = \int_0^{t_f} v^T(t)v(t)dt$ .

**Observation 5.1** Consider a system  $G:=(A,B,C,D)$  where  $A$  is asymptotically stable,  $x(0) = x_0$ ,  $t_f > 0$  and  $w(t) \in L_2$ . Define  $T_{zx_0} : x_0 \rightarrow z(t)$  and  $T_{zw} : w(t) \rightarrow z(t)$ . Then

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \|T_{zx_0}x_0 + T_{zw}w\|_2^2 - \gamma^2 \|w\|_2^2 \quad (113)$$

$$\leq \|T_{zx_0}x_0\|_2^2 + \|T_{zw}w\|_2^2 - \gamma^2 \|w\|_2^2 \quad (114)$$

$$\leq \|T_{zx_0}x_0\|_2^2 + (\|G\|_\infty^2 - \gamma^2) \|w\|_2^2 \quad (115)$$

The chain of inequalities can be verified using the norm-triangular inequality and inequality (21). Now it is important to differ between the cases  $x(0) = x_0 \neq 0$  and  $x(0) = 0$ . For  $x(0) = 0$  we recover the inequality (21.) When  $x(0) = x_0 \neq 0$  additional terms based on the system response to these initial conditions have to be considered. In general the following can be stated.

**Observation 5.2** ([49]) Consider a system  $G:=(A,B,C,D)$  where  $A$  is asymptotically stable,  $x(0) = x_0$ ,  $t_f \rightarrow \infty$  and  $w(t) \in L_2$ , then  $\|G\|_\infty < \gamma$  iff there is a  $\delta(x_0, w) > 0$  such that

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq -\delta(x_0, w) \|w\|_2^2 \quad (116)$$

Note that the above inequalities are valid for any  $x_0$  and any  $w(t)$  including the worst-case disturbance  $w_{worst}(t)$  in  $L_2$ . To append one of these inequalities with  $w(t) = w_{worst}(t)$  (by taking max of both sides) as a side constraint to an  $H_2$ -optimization problem the parameter  $\delta(x_0, w)$  has to be known beforehand.  $\delta(x_0, w)$  is not known apriori as it is a function of the disturbance  $w(t)$  and  $x_0$ . The defined function is still valuable in the  $H_\infty$ -sense due to properties discussed in the remaining section. To see this let us consider a maximization problem for finite time as follows.

**Definition 5.1** Consider the following optimization problem with  $G:=(A,B,C,D)$ ,  $A$  asymptotically stable,  $(A,B)$  controllable,  $(C,A)$  observable,  $x(0) = x_0$ ,  $t_f > 0$  and  $\gamma > \bar{\sigma}(D)$

$$J_1(w) = \max_{w \in L_2} \int_0^{t_f} \{z^T(t)z(t) - \gamma^2 w^T(t)w(t)\} dt \quad (117)$$

Note that the condition  $\gamma > \bar{\sigma}(D)$  is a necessary condition for the existence of a maximum. Applying standard Lagrange multiplier techniques, the above optimization problem results in a Two Point Boundary Problem (TPBVP):

**Theorem 5.1** Consider the problem defined in Definition 5.1 and for a finite  $t_f$ . Then the worst-case disturbance  $w_0(t)$  is given by

$$w_0(t) = (\gamma^2 I - D^T D)^{-1} [D^T C x(t) + B^T \lambda(t)] \quad (118)$$

subject to

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{pmatrix} = \bar{M} \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} \quad (119)$$

with boundary conditions

$$x(0) = x_0 \quad (120)$$

$$\lambda(t_f) = 0 \quad (121)$$

where

$$\bar{M} = \begin{pmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -\gamma^2 C^T S^{-1}C & -[A + BR^{-1}D^TC]^T \end{pmatrix}, \quad (122)$$

$R = (\gamma^2 I - D^T D)$ ,  $S = (\gamma^2 I - DD^T)$  and  $\lambda(t)$  being the Lagrange multiplier vector.

Note that this is an 'open-loop' characterization of  $w(t)$ , a closed-loop description of  $w_0(t)$  can be found in terms of the solution of a differential Riccati equation as follows.

**Theorem 5.2** *If a solution for the differential Riccati equation*

$$\begin{aligned} \dot{\bar{Z}}_1(t) + \bar{Z}_1(t)[A + BR^{-1}D^TC] + [A + BR^{-1}D^TC]^T \bar{Z}_1(t) \\ + \bar{Z}_1(t)BR^{-1}B^T \bar{Z}_1(t) + \gamma^2 C^T S^{-1}C = 0 \end{aligned} \quad (123)$$

with the boundary condition

$$\bar{Z}_1(t_f) = 0 \quad (124)$$

exists, then

$$J_1 = x_0^T \bar{Z}_1(0) x_0 \quad (125)$$

and

$$w_0(t) = (\gamma^2 I - D^T D)^{-1} [D^T C + B^T \bar{Z}_1(t)] x(t) \quad (126)$$

Equivalent results can be derived for the infinite-time case. We have

**Definition 5.2**

$$J_2(w) = \max_{w \in L_2} \lim_{t_f \rightarrow \infty} \int_0^{t_f} \{z^T(t)z(t) - \gamma^2 w^T(t)w(t)\} dt \quad (127)$$

with the same assumptions as in definition 5.1.

For the infinite time problem as  $t_f \rightarrow \infty$ , the differential Riccati equation will reduce to an algebraic Riccati equation.

**Theorem 5.3** *If a solution for the algebraic Riccati equation*

$$[A^T + C^T D R^{-1} B^T] Z_1 + Z_1 [A + BR^{-1} D^T C] + Z_1 B R^{-1} B^T Z_1 + \gamma^2 C^T S^{-1} C = 0 \quad (128)$$

exists and

$$A + B(\gamma^2 I - D^T D)^{-1} [D^T C + B^T Z_1] \quad (129)$$

is asymptotically stable, then

$$J_2 = x_0^T Z_1 x_0 \quad (130)$$

and

$$w_0(t) = (\gamma^2 I - D^T D)^{-1} [D^T C + B^T Z_1] x(t) \quad (131)$$



If a solution of this type exists then the associated TPBVP is also solvable. An important observation from this theorem is that the worst-case disturbance for  $t_f \rightarrow \infty$  is of a 'full state feedback type'. That is,  $w_0(t) = Kx(t)$  can be motivated.

Some very interesting connections between the  $H_\infty$ -norm and the optimization problem  $J_2$  are given by the Hamiltonian matrices  $M_\gamma$  (see equation (10)) and  $\bar{M}_\gamma$  and the associated Riccati equations. It is easy to verify that the following is true,

$$M_\gamma = T^{-1} \bar{M}_\gamma T \quad (132)$$

where

$$T = \begin{pmatrix} -\frac{1}{\sqrt{\gamma}}I & 0 \\ 0 & -\sqrt{\gamma}I \end{pmatrix} \quad (133)$$

Note that  $T$  is nonsingular and has no eigenvalue  $\lambda_i$  with  $\text{Re}(\lambda_i) = 0$  for any  $\gamma > 0$ . Thus we have the following lemma.

**Lemma 5.1** *Consider a system  $G := (A, B, C, D)$  with  $A$  asymptotically stable,  $(A, B)$  controllable,  $(C, A)$  observable and  $\gamma > \bar{\sigma}(D)$ , then the following statements are equivalent:*

1.  $\|G\|_\infty < \gamma$ .
2.  $[M_\gamma - j\omega I]$  nonsingular  $\forall \omega \in \mathbb{R}$ .
3.  $[\bar{M}_\gamma - j\omega I]$  nonsingular  $\forall \omega \in \mathbb{R}$ .
4. The ARE (13) has a unique real positive definite stabilizing solution  $X_1 = X_1^T$ .
5. The ARE (128) has a unique real positive definite stabilizing solution  $Z_1 = Z_1^T = \gamma X_1$ .
6.  $J_2 > 0$ .

The equivalence of these conditions is easily verified using previous lemmas and theorems. Note also that  $M_\gamma$  has the same eigenvalues as  $\bar{M}_\gamma$ . Furthermore, if the ARE (128) has a solution  $Z_1$  as characterized in Lemma 5.1, then this solution is unique and is the only solution that results in an asymptotically stable  $A + B(\gamma^2 I - D^T D)^{-1} [D^T C + B^T Z_1]$ . Furthermore, the matrix  $Z_1$  is the maximal solution of the ARE (128). That is, every other solution  $\hat{Z}_1$  of (128) satisfies  $Z_1 - \hat{Z}_1 > 0$ .

Assume that there exist solutions  $Z_1$  for the ARE (128) and  $\bar{Z}_1(t)$  for the differential Riccati equation (123), then the value of the cost function  $J_1(w)$  satisfies

$$J_1 = x_0^T \bar{Z}_1(0) x_0 = x_0^T Z_1 x_0 - x^T(t_f) Z_1 x(t_f) \quad (134)$$

Thus, it can be verified that the optimal value of  $J_1$  is monotonically increasing and  $Z_1 \geq \bar{Z}_1(0)$ . Hence  $Z_1$  represents an upper bound for  $J_1(w)$  in the limit as  $t_f \rightarrow \infty$ . With these preliminaries the following theorem can be stated:

**Theorem 5.4** ([53], [29]) *Consider a system  $G := (A, B, C, D)$  where  $A$  is asymptotically stable,  $(A, B)$  controllable,  $(C, A)$  observable,  $C^T D = 0$ ,  $B D^T = 0$  and  $\gamma > \bar{\sigma}(D)$ , then, for some  $x_0$*

$$\|G\|_\infty < \gamma \iff 0 < \max_w \lim_{t_f \rightarrow \infty} \int_0^{t_f} \{z^T(t)z(t) - \gamma^2 w^T(t)w(t)\} dt < \infty \quad (135)$$

The 'if and only if' relationship between  $\|G\|_\infty < \gamma$  and the nonnegativity of  $J_2$  has already been established in lemma (5.1), thus the only remaining part to be proven is the boundedness of  $J_2$ .

Willems in [53] stated a proof that connects  $\|G\|_\infty < \gamma$  and the finiteness of the above cost functional directly via the frequency inequality while Magierou [29] showed that the solution  $Z_1(t)$  of the differential Riccati equation (123) is monotonically increasing. An alternative proof based on the associated Hamiltonian is possible also. Thus, if there is no symmetric positive definite solution to the Riccati equation (128), then the solution  $Z_1(t)$  of (123) and thus the value of the cost functional  $J_2$  will grow unbounded for some  $x_0$ . Let us now define an alternative form of the above cost function, namely the cost function  $J_2$  with the worst-case disturbance  $w_0(t) = (\gamma^2 I - D^T D)^{-1} [D^T C + B^T \bar{K}] x(t)$  in place where the matrix  $\bar{K}$  is used as a design parameter in the determination of the worst-case disturbance. The orthogonality condition finally guarantees, that the worst-case disturbance can be generated by  $\bar{K}$ .

**Definition 5.3** Consider a system  $G := (A, B, C, D)$  where  $A$  is asymptotically stable,  $(A, B)$  controllable,  $(C, A)$  observable and  $\gamma > \bar{\sigma}(D)$ . We define

$$J_3(\bar{K}) = \max_{\bar{K}} \lim_{t_f \rightarrow \infty} \text{Trace} \left\{ \int_0^{t_f} e^{F^T t} [\gamma^2 C^T S^{-1} C - \bar{K} B R^{-1} B^T \bar{K}] e^{F t} dt \right\} \quad (136)$$

where  $R$  and  $S$  are defined in equation (122) and

$$F = A + B R^{-1} [D^T C + B^T \bar{K}] \quad (137)$$

Note that  $J_3(\bar{K})$  is just  $J_2(w)$  with the parametrization of  $w(t) = (\gamma^2 I - D^T D)^{-1} [D^T C + B^T \bar{K}] x(t)$ . With the assumption  $\gamma > \bar{\sigma}(D)$  we assure that  $R$  and  $S$  are positive definite matrices. We now know that, if  $\|G\|_\infty < \gamma$ , the unique maximizing  $\bar{K} = Z_1$  is positive definite, symmetric and the closed loop system is asymptotically stable, in this case  $0 < J_2(Z_1) = J_3(\bar{K}) < \infty$ . If we consider the case  $\bar{\sigma}(D) < \gamma \leq \|G\|_\infty$  then the ARE (128) indicates that there is no positive definite  $Z_1$  that 'stabilizes' the system (there is no solution of ARE (128) at all that results in a stable closed-loop system at all). That is, there is no symmetric positive definite  $\bar{K}$  that solves the associated Riccati equation such that  $F$  is asymptotically stable. If we look at the maximization problem for  $J_3(\bar{K})$  it can be verified that for  $\bar{K} = \varepsilon I$  (for some  $\varepsilon > 0$ )  $J_3(\bar{K}) > 0$  can be achieved. Thus we can conclude that  $J_3(\bar{K}) > 0$  for any maximizing  $\bar{K}$  as long as we assume  $\gamma > \bar{\sigma}(D)$ . Furthermore, for  $\bar{\sigma}(D) < \gamma \leq \|G\|_\infty$ , we know that the maximizing  $\bar{K}$  will result in an unbounded cost function value  $J_3(\bar{K})$  for  $t_f \rightarrow \infty$ . Note, that the cost function  $J_3(\bar{K})$  corresponds to an initial condition  $x_0$  with  $x_0 x_0^T = I$ . Such an initial condition does not exist, this assumption has its justification from stochastic interpretation in terms of covariance matrices. It can be seen that  $\bar{K}$  will attempt to destabilize  $F$  by being 'large', at the same time  $\bar{K}$  must be 'small' enough to not excite this 'destabilized' mode of  $F$  with a negative eigenvalue of  $[\gamma^2 C^T S^{-1} C - \bar{K} B R^{-1} B^T \bar{K}]$ . This would lead to a negative unbounded cost function value which is not in the interest of a maximizing strategy. We also could use a parametrization  $w_0(t) = K x(t)$ . With this parametrization of the worst-case disturbance for the  $H_\infty$ -bound problem we arrive at a cost function  $\bar{J}_3(K)$  similar to  $J_3(\bar{K})$ . Namely

$$\bar{J}_3(K) = \max_K \lim_{t_f \rightarrow \infty} \text{Trace} \left\{ \int_0^{t_f} e^{G^T t} [(C + D K)^T (C + D K) - \gamma^2 K^T K] e^{G t} dt \right\} \quad (138)$$

where  $G = A + BK$ .

Similar observations can be made about this cost function. For  $K = 0$ ,  $\bar{J}_3(K) \geq 0$ . Thus every maximizing  $K$  once more will yield a non-negative cost. However, it will be more convenient to adopt a parametrization as in (131), as this parametrization will allow us a better comparison with Riccati-based approaches.

The additional term  $\int_0^{t_f} [-\gamma^2 w^T(t)w(t)]dt$  in the cost function  $J_2(w)$  plays a similar role as the term  $\int_0^{t_f} [\frac{1}{\rho} u^T(t)u(t)]dt$  in the LQ-design. As  $\rho \rightarrow \infty$ , the 'cheap' control case is approached in LQ-design while  $\gamma \leq \|G_{zw}\|_\infty$  is equivalent to 'cheap disturbances', meaning that the disturbances  $w$  can have unbounded power. The worst-case  $H_\infty$  disturbance  $w$  is necessarily of bounded power if  $\|G_{zw}\|_\infty < \gamma$  so that this cost function actually represents  $\max_w \lim_{t_f \rightarrow \infty} \int_0^{t_f} z^T(t)z(t)dt$  for  $\|G_{zw}\|_\infty < \gamma$  if we iterate on  $\gamma$  to achieve  $\|w_0\|_2 = 1$ . If the solution exists, then furthermore  $w_0 \in L_2$ .

### Example:

Consider the following system  $G := (A, B, C, D)$  with

$$A = \begin{pmatrix} -0.0168 & 0.1121 & 0.0003 & -0.5608 \\ -0.0164 & -0.7771 & 0.9945 & 0.0015 \\ -0.0417 & -3.6595 & -0.9544 & 0 \\ 0 & 0 & 1.0000 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -0.0243 \\ -0.0634 \\ -3.6942 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad D = 0,$$

then it is easy to verify that the conditions in theorem 5.4 are satisfied. The  $\infty$ -norm of  $G$  can be computed to be  $\|G\|_\infty^2 = 2.9903$ . The following plot then shows the value of the cost function (136) in Definition 5.3 over the finite time  $t_f$  for three different values of  $\gamma$ . It is easily seen that, for  $\gamma < \|G\|_\infty$ , the plot diverges to large values for large  $t_f$  and to infinite values as  $t_f \rightarrow \infty$ , while the cost remains bounded and converges to a steady-state value for large  $t_f$  if  $\gamma$  is chosen such that  $\gamma > \|G\|_\infty$ .

The  $H_\infty$ -control problem is in general a min-max problem. We minimize a cost function with respect to the controller parameters while maximizing the cost function with respect to the disturbances. To see the connection between this min-max  $H_\infty$ -control problem and the cost function in this section let us look to a game theoretical problem associated with the cost function. To shorten this presentation only the case for infinite  $t_f$  is considered. For finite  $t_f$  similar results can be developed. For this purpose define a system as follows.

$$\mathbf{G} := \begin{cases} \dot{x} &= Ax + Bw + \Gamma u, & x(0) = x_0 \\ z &= Cx + Dw + \Omega u \end{cases} \quad (139)$$

where  $x(t)$  represents the state of the system,  $u(t)$  the control input and  $w(t)$  the disturbance. Now a game problem can be formulated as follows.

**Definition 5.4** Consider a system described in (139) with  $(A, B)$  controllable,  $(A, \Gamma)$  controllable,  $(C, A)$  observable,  $D = 0$ ,  $\Omega^T C = 0$ ,  $\Omega^T \Omega > 0$  and define

$$J_4(u, w) = \min_{u \in L_2} \max_{w \in L_2} \lim_{t_f \rightarrow \infty} \int_0^{t_f} \{z^T(t)z(t) - \gamma^2 w^T(t)w(t)\} dt \quad (140)$$

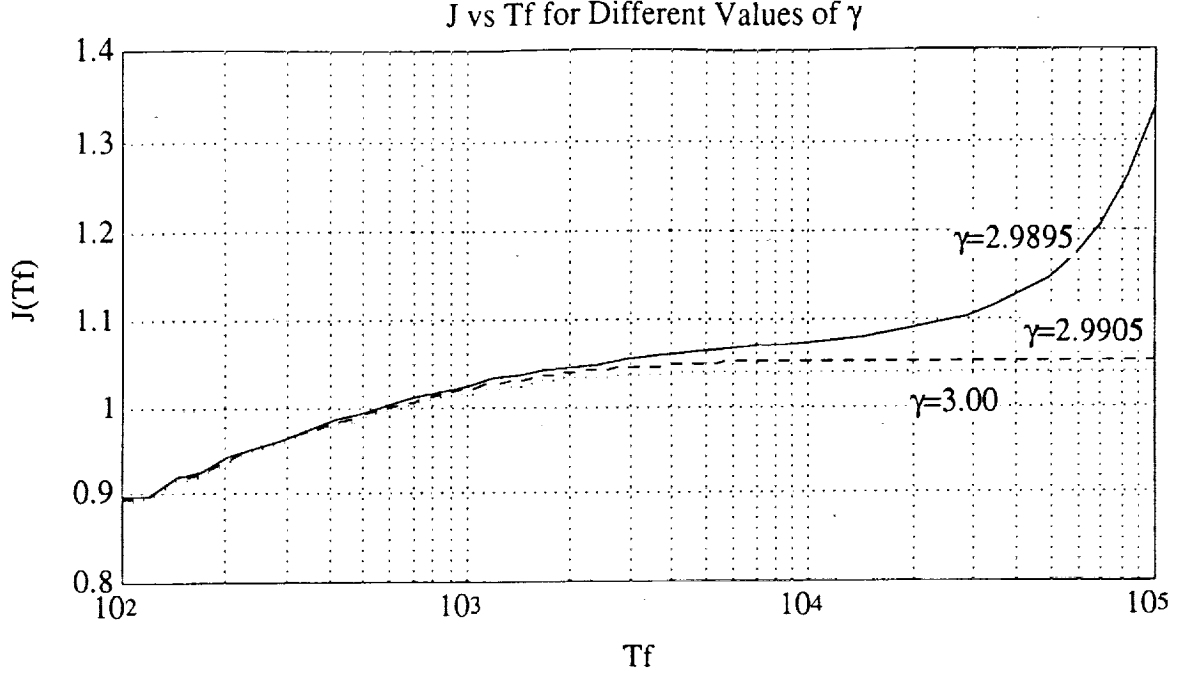


Figure 1: Worst-Case Design

Note that the assumption  $D = 0$  is not restrictive, it can be easily removed. This assumption is only made to facilitate our presentation (for a complete treatment see [24]). Without the assumption  $\Omega^T \Omega > 0$  the problem is a singular game problem. Let us skip here the associated Two Point Boundary Problem and examine directly the associated Riccati equation for  $t_f \rightarrow \infty$ .

**Theorem 5.5** ([35] [29] [53]) *In the limit as  $t_f \rightarrow \infty$  the min-max problem stated in Definition 5.4 with a plant (139) has a finite solution for some  $x_0$  if and only if the ARE*

$$Z_2[A - \Gamma(\Omega^T \Omega)^{-1} \Omega^T C] + [A - \Gamma(\Omega^T \Omega)^{-1} \Omega^T C]^T Z_2 - Z_2[\Gamma(\Omega^T \Omega)^{-1} \Gamma^T - \frac{1}{\gamma^2} B B^T] Z_2 + C^T C = 0 \quad (141)$$

has a positive semi-definite solution  $Z_2$  such that

$$A - \Gamma(\Omega^T \Omega)^{-1} [\Omega^T C + \Gamma^T Z_2] + \frac{1}{\gamma^2} B B^T Z_2$$

is asymptotically stable, then the optimal strategies are given as

$$u_{opt}(t) = K_u x(t) = -(\Omega^T \Omega)^{-1} [\Omega^T C + \Gamma^T Z_2] x(t) \quad (142)$$

$$w_{opt}(t) = K_w x(t) = \frac{1}{\gamma^2} B^T Z_2 x(t) \quad (143)$$

and the value of the game is given by

$$J_4(u_{opt}, w_{opt}) = x_0^T Z_2 x_0 > 0 \quad (144)$$

Furthermore, in this case  $w_{opt}(t) \in L_2$ .

The resulting ARE is a so-called game-theoretical ARE, characterized by the sign-indefinite term  $[\Gamma(\Omega^T\Omega)^{-1}\Gamma^T - \frac{1}{\gamma^2}BB^T]$ . The proof is based on the same references as in Theorem 5.4. Connection between this game theoretical problem and the  $H_\infty$ -control problem has been pointed out by - among others - Tadmor in [49]. It can be verified by comparing ARE (141) with the ARE in Theorem 2 of [13]. Let  $\gamma_0$  denote the smallest  $H_\infty$ -norm from  $w$  to  $z$  in (139) achievable with pure state feedback, that is  $u(t) = K_u x(t)$ . Let  $T_{zw}$  denote the closed-loop transfer function between  $w$  and  $z$  in (139) with  $u = K_u x$  in place, then

**Theorem 5.6** ([49], [13]) *Under the assumptions made in [13] and  $\Omega^T\Omega > 0$ , there exists a state feedback control  $u(t) = K_u x(t)$  such that  $\gamma > \gamma_0$  (that is  $\min_{K_u} \|T_{zw}\|_\infty = \gamma_0 < \gamma$ ) if and only if the ARE (141) has a symmetric positive definite solution  $Z_2$  such that  $A - \Gamma(\Omega^T\Omega)^{-1}[\Omega^TC + \Gamma^TZ_2] + \frac{1}{\gamma^2}BB^TZ_2$  is asymptotically stable.*

Thus, a finite value of the above game, and hence with  $J_4(u, w)$  being bounded implies that there is a state feedback matrix that stabilizes the system and achieves  $\|T_{zw}\|_\infty < \gamma$ . Furthermore, the worst-case disturbance can be of the form  $w_{opt}(t) = Kx(t)$ . If we consider static state feedback controllers, then  $J_4(u, w)$  being unbounded for a chosen  $\gamma$  implies that there is no state feedback matrix that achieves the bound  $\|T_{zw}\|_\infty < \gamma$ . This, in turn, implies  $\gamma \leq \gamma_0$ . If a general compensator structure is used, then unboundedness of  $J_4(u, w)$  implies that the given controller cannot satisfy the bound  $\|T_{zw}\|_\infty < \gamma$ . It is obvious now that the cost functional considered in this section represents a useful tool to incorporate an  $H_\infty$ -constraint into a  $H_2$ -cost function.

## 6 A LQ-Type Cost Function for Mixed $H_2/H_\infty$ -Control Design

The previous section motivates a parametrization of the worst-case  $H_\infty$ -disturbance in the form of a state feedback  $w(t) = Kx(t)$ . It should be noted, that other parametrizations are also possible. We could, for example, assume that  $w$  is generated by an independent linear system (an open-loop parametrization) or a factorization  $w(t) = \bar{w}_0 e^{j\omega t}$  in the definition of a frequency-domain cost function. The state-feedback characterization, however, has proven to provide many desirable connections to the  $H_\infty$ -bound problem.

In this section we propose a LQ-cost function that represents the mixed  $H_2/H_\infty$ -bound control problem.

### 6.1 The One Disturbance/Two Criterion Case

The system under consideration represents the most general case of the one disturbance input/two criterion output case. Let

$$\Sigma_S^{1/2} := \begin{cases} \dot{x} &= Ax + B_1w + B_3u, & x(0) = x_0 \\ z_2 &= C_1x + D_{11}w + D_{13}u \\ z_\infty &= C_2x + D_{21}w + D_{23}u \\ y &= C_3x + D_{31}w + D_{33}u \end{cases} \quad (145)$$

where it is assumed that the system uncertainties are all lumped into one  $\Delta$ -block already, namely  $w = \Delta z_\infty$  with  $\|\Delta\|_\infty < \frac{1}{\gamma+\alpha}$  where  $\alpha$  is some positive constant. Note that the small gain theorem then provides robust stability as long as the  $H_\infty$ -norm of the transfer function from  $w$  and  $z_\infty$  is smaller than  $\gamma$ . Furthermore the following assumptions are made:

**A1:**  $(A, B_1)$  and  $(A, B_3)$  are controllable pairs,

**A2:**  $(C_1, A)$ ,  $(C_2, A)$  and  $(C_3, A)$  are observable pairs.

The compensator is assumed to be proper with the general structure,

$$\Sigma_c := \begin{cases} \dot{x}_c &= A_c x_c + B_c y, & x_c(0) = x_{c0} \\ u &= C_c x_c + D_c y \end{cases} \quad (146)$$

For the  $H_\infty$ -part of our cost function the disturbance  $w$  is parametrized in the form

$$w = K_1 x + K_2 x_c = K \begin{pmatrix} x \\ x_c \end{pmatrix} \quad (147)$$

Define furthermore the two subsystems,

$$\Sigma_2 := \begin{cases} \dot{x} &= Ax + B_1 w + B_3 u, & x(0) = x_0 \\ z_2 &= C_1 x + D_{11} w + D_{13} u \\ y &= C_3 x + D_{31} w + D_{33} u \end{cases} \quad (148)$$

$$\Sigma_\infty := \begin{cases} \dot{x} &= Ax + B_1 w + B_3 u, & x(0) = x_0 \\ z_\infty &= C_2 x + D_{21} w + D_{23} u \\ y &= C_3 x + D_{31} w + D_{33} u \end{cases} \quad (149)$$

Then the closed-loop system with the compensator in place has the following form

$$\Sigma_{\text{Scl}}^{1/2} := \begin{cases} \dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} w, & x_{cl}(0) = x_{cl0} \\ z_2 &= C_{1cl} x_{cl} + D_{11cl} w \\ z_\infty &= C_{2cl} x_{cl} + D_{21cl} w \end{cases} \quad (150)$$

where

$$\begin{aligned} x_{cl} &= \begin{pmatrix} x \\ x_c \end{pmatrix} \\ A_{cl} &= \begin{pmatrix} A + B_3(I - D_c D_{33})^{-1} D_c C_3 & B_3(I - D_c D_{33})^{-1} C_c \\ B_c C_3 + B_c D_{33}(I - D_c D_{33})^{-1} D_c C_3 & A_c + B_c D_{33}(I - D_c D_{33})^{-1} C_c \end{pmatrix} \\ B_{cl} &= \begin{pmatrix} B_1 + B_3(I - D_c D_{33})^{-1} D_c D_{31} \\ D_{31} + D_{33}(I - D_c D_{33})^{-1} D_c D_{31} \end{pmatrix} \\ C_{1cl} &= \begin{pmatrix} C_1 + D_{13}(I - D_c D_{33})^{-1} D_c C_3 & D_{13}(I - D_c D_{33})^{-1} C_c \end{pmatrix} \\ D_{11cl} &= \begin{pmatrix} D_{11} + D_{13}(I - D_c D_{33})^{-1} D_c D_{31} \end{pmatrix} \end{aligned}$$

$$C_{2cl} = \begin{pmatrix} C_2 + D_{23}(I - D_c D_{33})^{-1} D_c C_3 & D_{23}(I - D_c D_{33})^{-1} C_c \end{pmatrix}$$

$$D_{21cl} = \begin{pmatrix} D_{21} + D_{23}(I - D_c D_{33})^{-1} D_c D_{31} \end{pmatrix}$$

Let  $C_0$  denote a matrix

$$C_0 = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix}$$

that contains all the compensator matrices. Now the cost function can be defined to represent the mixed  $H_2/H_\infty$ -criterion. To do that, we define the two separate cost functions  $J^2(C_0, t_f)$ ,  $J^\infty(C_0, K, t_f)$  as follows.

**Definition 6.1** Define

$$J^2(C_0, t_f) = \mathcal{E}[z_2^T(t_f)Qz_2(t_f) + u^T(t_f)R_1u(t_f)] \quad (151)$$

subject to the system  $\Sigma_2$ , the compensator  $\Sigma_c$ ,  $w(t)$  white noise with unit spectral density,  $Q = Q^T > 0$ ,  $R_1 = R_1^T > 0$  and the constraint  $|C_{0ij}| \leq M_c \quad \forall i, j$ , where  $M_c$  is an upper bound for every entry in  $C_0$ .

**Definition 6.2** Define

$$J^\infty(C_0, w, t_f) = J^\infty(C_0, K, t_f) = \int_0^{t_f} [z_\infty^T(t)z_\infty(t) - \gamma^2 w^T(t)w(t)]dt \quad (152)$$

subject to the system  $\Sigma_\infty$  and the compensator  $\Sigma_c$ . If a certain controller  $C_0$  is given, perform the preliminary transformation  $w(t) = (\gamma^2 I - D_{11cl}^T D_{11cl})^{-1} [D_{11cl}^T C_{1cl}x(t) + B_{cl}^T \bar{w}(t)]$ ,  $\bar{w}(t) = Kx_{cl}(t)$ ,  $K = K^T > 0$  and  $|K_{ij}| \leq M_K \quad \forall i, j$ , where  $M_K$  is an upper bound on all entries of  $K$ .

With all these preliminaries we now can finally state the cost function:

**Definition 6.3** Assume a system  $\Sigma_S^{1/2}$  and a compensator  $\Sigma_c$ . Then the cost function for a mixed  $H_2/H_\infty$ -design is defined as follows,

$$J^{2/\infty}(C_0, K) = \min_{C_0} \{ \beta [\lim_{t_f \rightarrow \infty} J^2(C_0, t_f)] + (1 - \beta) [\max_K \lim_{t_f \rightarrow \infty} J^\infty(C_0, K, t_f)] \} \quad (153)$$

subject to  $D_{11cl} = 0$  for boundedness of  $J^2(C_0, \infty)$ ,

$\bar{\sigma}(D_{21cl}) < \gamma$ ,

$(I - D_c D_{33})$  is nonsingular for well posedness,

$[(C_{2cl} + D_{21cl}K)^T(C_{2cl} + D_{21cl}K) - \gamma^2 K^T K] \geq 0 \quad \forall C_0$  for non-negativity of the cost function,

$(A_{cl}, B_{cl})$  is a controllable pair,

$(C_{1cl}, A_{cl})$  and  $(C_{2cl}, A_{cl})$  are observable pairs,  $C_{2cl}^T D_{21cl} = 0$ ,  $B_{cl} D_{21cl}^T = 0$

and  $\beta \in [0, 1]$ .

### 6.1.1 Interpretation of the Cost Function and Comments

1. In general the cost function represents the weighted sum of a weighted  $H_2$ -criterion  $J^2(C_0, t_f)$  and a  $H_\infty$ -bound criterion  $J^\infty(C_0, K, t_f)$ . That is, having chosen a  $\gamma$ , the design based on this cost function attempts to minimize the  $H_2$ -norm from  $w$  to  $z_2$  while satisfying the  $H_\infty$ -norm-bound from  $w$  to  $z_\infty$  by 'minimizing'  $J^\infty(C_0, K)$  to a finite value (in the limit for  $t_f \rightarrow \infty$ ). Note, that - in the limit as  $t_f \rightarrow \infty$  -  $J^{2/\infty}(C_0, K)$  will be unbounded if either the design yields an unstable system or the  $H_\infty$ -bound is violated. If  $J^{2/\infty}(C_0, K)$  is finite, we know that the design achieved a stable closed-loop system with an  $H_\infty$ -norm from  $w$  to  $z_\infty$  larger than  $\gamma_0$ , the minimal  $H_\infty$ -norm achievable with state feedback. The algorithm proposed in the next chapter will give more insights into this problem when general compensator structures are considered.
2. The set of all feasible compensators can be characterized as follows:  
 "A compensator  $\Sigma_c$  is feasible if the compensator yields a finite value for  $J^{2/\infty}(C_0, K)$ ."  
 This characterization can be based on Theorem 5.5 or on the algorithm used for the actual computation of the compensator, namely a sequence of minimizations and maximizations. The set of feasible compensators can then be motivated by Theorem 5.4.
3. The cost function as posed represents a 'simultaneous mixed  $H_2/H_\infty$ -bound-approach'. That means, it represents an unconstrained optimization problem (unconstrained in the  $H_\infty$ -sense), as both objectives are explicitly contained in the cost function. The design objectives can be summarized in the following form:  
 "Find a controller that internally stabilizes the system, that furthermore minimizes the 2-norm from  $w$  to  $z_2$  and satisfies an  $H_\infty$ -norm bound from  $w$  to  $z_\infty$ ."  
 It can be verified now that the set of feasible compensators as defined above satisfies these design objectives.
4. The assumption  $D_{11cl} = 0$  is necessary for a finite  $H_2$ -norm from  $w$  to  $z_2$ .
5.  $\bar{\sigma}(D_{21cl}) < \gamma$  guarantees that the cost function  $\max_K \lim_{t_f \rightarrow \infty} J^\infty(C_0, K, t_f)$  is positive for a given  $C_0$ , it furthermore is a necessary condition for the existence of a maximum for  $\max_w J^\infty(C_0, w, t_f)$ . It is easily seen that in the case of a strictly proper compensator it is necessary for  $\|G_{z_\infty w}\|_\infty < \gamma$  that  $\bar{\sigma}(D_{21}) < \gamma$ .
6. The observability and controllability assumptions **A1** and **A2** as well as the orthogonality assumptions in definition 6.3 assure that there is a  $x_{cl}(0)$  such that the cost function  $J^\infty(C_0, K, t_f)$  will grow unbounded when the  $H_\infty$ -bound is violated (unbounded in the limit as  $t_f \rightarrow \infty$ ). The controllability assumptions are also necessary for the system to be stabilizable by  $u$  and disturbable by  $w$  while the observability condition assures that there are no unstable hidden modes unaccounted for in the cost function.
7. The assumption  $([C_{2cl} + D_{21cl}K]^T[C_{2cl} + D_{21cl}K] - \gamma^2 K^T K) \geq 0 \quad \forall C_0$  is a sufficient condition for  $\min_{C_0} \lim_{t_f \rightarrow \infty} J^\infty(C_0, K, t_f)$  to be non-negative for a given  $K$ . Without this assumption the minimization of  $J^\infty(C_0, K, t_f)$  might result in a negative infinite function value. Cases can be constructed, where only a small adjustment of the controller parameters results in a negative infinite cost (without really changing the dynamics of the compensator during the minimizations). Since



our approach is gradient based (a sequence of minimizations and maximizations) the case could arise where the cost function  $J^{2/\infty}(C_0, K)$  alternates indefinitely between a large negative cost (when the minimization is performed) and a large positive cost (when the maximization is performed) without convergence. The assumption  $([C_{2cl} + D_{21cl}K]^T[C_{2cl} + D_{21cl}K] - \gamma^2 K^T K) \geq 0 \quad \forall C_0$  excludes this case.

8.  $J^{2/\infty}(C_0, K)$  is continuous in  $K$  as well as in  $C_0$  as long as  $(I - D_c D_{33})$  is nonsingular. If continuity problems arise in the algorithm we can always assume  $D_{33} = 0$  for the design and then reincorporate  $D_{33}$  if the final design for the plant with  $D_{33} = 0$  is well posed (see [66] section 5.5.2).
9. The restriction on the entries of  $C_0$  and  $K$  guarantees that the domains for  $C_0$  and  $K$  are closed and bounded. This restriction also avoids the 'infinite gain case'. That is, no element of  $C_0$  or  $K$  can grow unbounded.
10.  $\beta$  is a weighting factor that allows a mixing of the  $H_\infty$ -bound- and  $H_2$ -criteria. In the limiting cases ( $\beta = 1$  and  $\beta = 0$ ) the implications for the optimization problem are obvious, namely the reduction of the cost function to either a pure  $H_2$ -problem or a pure  $H_\infty$ -bound problem. However, as pointed out in the results due to Doyle, Zhou and Khargonekar, the mixed  $H_2/H_\infty$ -problem as stated is solvable if and only if the pure  $H_\infty$ -bound problem has a solution. Thus, at this point the implications of  $\beta \in (0, 1)$  are not completely clear, practical designs and their dependence on  $\beta$  will be considered.
11. The cost function allows an optimization of the  $H_2$ -criterion over noise distributions other than white noise by using shaping filters.
12. If an additional  $\gamma$ -iteration is included in the algorithm, the true mixed  $H_2/H_\infty$ -problem can also be addressed. Furthermore, if structural knowledge about  $\Delta$  is known the  $\mu$ -synthesis problem can be addressed by introducing appropriate unitary scaling matrices (see e.g. [12]).
13. The cost function  $J^{2/\infty}(C_0, K)$  is not concave in  $K$ , nor is it convex in  $C_0$ . At least no proof has been found so far that it is concave in  $K$ . This raises the question if there is a different parametrization of the worst case  $H_\infty$ -disturbance that forms a concave optimization problem.

## 6.2 The Two Disturbance/One Criterion Case

The above considerations can be applied to the two disturbance input/one criterion output case. Here we consider the most general system with two disturbance inputs and one criterion output,

$$\Sigma_S^{2/1} := \begin{cases} \dot{x} &= Ax + B_1 w_2 + B_2 w_\infty + B_3 u, & x(0) = x_0 \\ z &= C_1 x + D_{11} w_2 + D_{12} w_\infty + D_{13} u \\ y &= C_3 x + D_{31} w_2 + D_{32} w_\infty + D_{33} u \end{cases} \quad (154)$$

with the additional assumptions

**A1'**:  $(A, B_i)$  for  $i=1,2,3$  form controllable pairs,

**A2'**:  $(C_1, A)$  and  $(C_3, A)$  form observable pairs.

The compensator for this case is assumed to be of the same form as in the one disturbance/two criterion case. Forming the closed loop system with the compensator in place yields

$$\Sigma_{\text{Scl}}^{2/1} := \begin{cases} \dot{\bar{x}}_{cl} &= \bar{A}_{cl}\bar{x}_{cl} + \bar{B}_{1cl}w_2 + \bar{B}_{2cl}w_\infty, & \bar{x}_{cl}(0) = \bar{x}_{cl0} \\ z &= \bar{C}_{1cl}\bar{x}_{cl} + \bar{D}_{11cl}w_2 + \bar{D}_{12cl}w_\infty \end{cases} \quad (155)$$

where

$$\begin{aligned} \bar{x}_{cl} &= \begin{pmatrix} x \\ x_c \end{pmatrix} \\ \bar{A}_{cl} &= \begin{pmatrix} A + B_3(I - D_c D_{33})^{-1} D_c C_3 & B_3(I - D_c D_{33})^{-1} C_c \\ \bar{B}_c C_3 + B_2 D_{33}(I - D_c D_{33})^{-1} D_c C_3 & A_c + B_2 D_{33}(I - D_c D_{33})^{-1} C_c \end{pmatrix} \\ \bar{B}_{1cl} &= \begin{pmatrix} B_1 + B_3(I - D_c D_{33})^{-1} D_c D_{31} \\ \bar{B}_c D_{31} + B_2 D_{33}(I - D_c D_{33})^{-1} D_c D_{31} \end{pmatrix} \\ \bar{B}_{2cl} &= \begin{pmatrix} B_2 + B_3(I - D_c D_{33})^{-1} D_c D_{32} \\ \bar{B}_c D_{32} + B_2 D_{33}(I - D_c D_{33})^{-1} D_c D_{32} \end{pmatrix} \\ \bar{C}_{1cl} &= \begin{pmatrix} C_1 + D_{13}(I - D_c D_{33})^{-1} D_c C_3 & D_{13}(I - D_c D_{33})^{-1} C_c \end{pmatrix} \\ \bar{D}_{11cl} &= \begin{pmatrix} D_{11} + D_{13}(I - D_c D_{33})^{-1} D_c D_{31} \end{pmatrix} \\ \bar{D}_{12cl} &= \begin{pmatrix} D_{12} + D_{13}(I - D_c D_{33})^{-1} D_c D_{32} \end{pmatrix} \end{aligned}$$

The cost function for mixed  $H_2/H_\infty$ -control is similar to the one defined above. However, here  $z$  plays the dual role that  $w$  played in the one disturbance/two criterion case.

**Definition 6.4** Consider the system  $\Sigma_S^{2/1}$  and the compensator  $\Sigma_c$ . Then the cost function for a mixed  $H_2/H_\infty$ -design is defined as follows

$$\bar{J}^{2/\infty}(C_0, K) = \min_{C_0} \max_{w_2, K} \lim_{t_f \rightarrow \infty} \int_0^{t_f} [z^T(t)z(t) - \gamma^2 w_\infty^T(t)w_\infty(t)] dt \quad (156)$$

subject to  $\bar{D}_{11cl} = 0$ ,  $\bar{\sigma}(\bar{D}_{12cl}) < \gamma$ ,  $(I - D_c D_{33})$  is nonsingular and  $w_\infty(t) = K\bar{x}_{cl}(t)$  and  $w_2(t)$  white noise with unit spectral density.

Note that this definition represents the 'key cost function' for our design approach to the mixed  $H_2/H_\infty$  problem. Additional assumptions have to be included in order to avoid negativity of the cost function or unobservability/uncontrollability. This cost function has similar interpretations as those given in the one disturbance/two criterion case and are omitted here.

## 7 Algorithm For The Proposed Mixed $H_2/H_\infty$ -Control Design

As mentioned in the introduction, the setting is that of a finite time with an LQ-type cost function for the mixed  $H_2/H_\infty$ -optimization. This finite time setting has the advantage that all defined LQ-type cost functions are well defined. In the limit as  $t_f \rightarrow \infty$ , time domain functions can only be expressed in terms of Lyapunov and Riccati equations. This formulation has the 'advantage' that, in the steady-state form, it recovers the problem associated with Riccati-based approaches. The approach taken here follows [23] in which a certain pre-specified compensator structure is chosen (not necessarily of the same order as the plant). Starting at a small  $t_f$ , this compensator structure is optimized with respect to the worst-case disturbances. Once an optimized solution for the compensator has been found,  $t_f$  is increased and the process repeated. The design procedure eventually settles to a  $t_f$  representative of the steady-state condition. Even if the final result of the algorithm does not yield a feasible compensator, we still can make statements about the resulting closed-loop system based on the individual values of  $J^\infty(C_0, K, t_f)$  and  $J^2(C_0, K, t_f)$  and their 'time-histories'.

Attention in this section is restricted to the one disturbance/two criterion case, an equivalent algorithm can be defined for the dual case of two disturbances and one criterion output. Note, that the problem

$$\min_{C_0} \max_K \lim_{t_f \rightarrow \infty} J^\infty(C_0, K, t_f) \quad (157)$$

as stated in Definition 6.2 is a true min-max problem added to the pure  $H_2$ -problem. For the  $H_2$ -problem we already know that the worst-case disturbance is white noise so that the  $H_2$ -part of the problem is actually only a min-problem.

A complete derivation of the pure  $H_\infty$ -bound problem in terms of game theory has been presented recently by Rhee and Speyer in [42]. Solutions are given in terms of two ARE's (the same as derived in the DGKF paper). A complete analytical treatment of the mixed  $H_2/H_\infty$ -bound problem in game theoretical terms, however, is still missing.

As discussed earlier, there is no guarantee for the existence of a saddle point solution for the min-max problem as posed by  $J^{2/\infty}(C_0, K)$ . The algorithm used here is the algorithm as suggested by Salmon (see Chapter 2) that is modified to suit our problem formulation. It searches for min-max optimal strategies in the sense of definition 2.14. The assumptions made in the last chapter assure that the optimization problem is continuous in  $C_0$  and  $K$  with closed, bounded domains. No further assumptions are made. The algorithm and a flow chart for the actual design are outlined as follows.

### The Algorithm:

#### 1. Initialization:

- Choose initial  $t_{f0}$ ,  $C_0$  (not necessarily stabilizing),  $x_0$ ,  $x_c(0)$ ,  $\gamma$ ,  $\beta$ ,  $Q$  and  $R_1$ .
- Select an initial set  $K^1 = \{K^1, K^2, \dots\}$ .
- Set  $n = 1$ ,  $i = 1$ .

2.  $n^{th}$  min-max iteration for  $t_f = t_{fi}$ :

If  $A_{cl}$  is unstable let  $\hat{\beta} = 1$ , otherwise let  $\hat{\beta} = \beta$ .

Perform a minimization with respect to  $C_0$  as stated in Definition 6.3 for all  $K \in \mathcal{K}^n$ :

$$\min_{C_0} \{ \hat{\beta} [J^2(C_0, t_{fi})] + (1 - \hat{\beta}) [\max_{K \in \mathcal{K}^n} J^\infty(C_0, K, t_{fi}) dt] \} \quad (158)$$

Let the minimizing compensator be denoted by  $C_0^n$ .

Compute  $J_n^m(t_{fi}) = \max_{K \in \mathcal{K}^n} J^{2/\infty}(C_0^n, K, t_{fi})$

Perform a maximization with respect to  $K$  as stated in Definition 6.2

$$\max_K J^\infty(C_0^n, K, t_{fi}) \quad (159)$$

and let the maximizing  $K$  be denoted by  $K^n$ .

Compute  $J_n^M(t_{fi}) = J^{2/\infty}(C_0^n, K^n, t_{fi})$ .

Form  $\mathcal{K}^{n+1} = \mathcal{K}^n \cup \{K^n\}$ .

3. **Termination criterion for  $t_f = t_{fi}$ :**

If  $|J_n^m(t_{fi}) - J_{n-1}^m(t_{fi})| \geq \varepsilon$ , let  $n = n + 1$ ,  $i = i$ , go to step 2.

If  $|J_n^m(t_{fi}) - J_{n-1}^m(t_{fi})| < \varepsilon$  compute  $J_i^{2/\infty}(t_{fi}) = \frac{1}{2} \{J_n^M(t_{fi}) + J_n^m(t_{fi})\}$  and continue at step 4.

4. **General termination criterion:**

If  $|J_i^{2/\infty}(t_{fi}) - J_{i-1}^{2/\infty}(t_{f(i-1)})| \geq \varepsilon$  let  $t_{f(i+1)} = t_{fi} + \Delta t_f$ ,  $i = i + 1$ ,  $n = 1$ ,  $\mathcal{K}^1 = \mathcal{K}^n$  and continue at step 2.

If  $|J_i^{2/\infty}(t_{fi}) - J_{i-1}^{2/\infty}(t_{f(i-1)})| < \varepsilon$  stop.

**Comments:**

1. Pure continuity of  $J^{2/\infty}(C_0, K)$  in  $C_0$  and  $K$  is sufficient for  $J_n^M(t_{fi})$  to be unbounded as  $t_f \rightarrow \infty$  if the  $H_\infty$ -bound is violated and for  $J_n^m(t_{fi})$  to be a monotonically increasing function of  $n$  (the latter fact is ascertained by the optimization over an increasing set  $\mathcal{K}$ ). From the considerations in Chapter 2 we also know that  $J_n^M(t_{fi})$  is an upper limit for  $\min_{C_0} \max_K J^{2/\infty}(C_0, K, t_{fi})$  and  $J_n^m(t_{fi})$  is a lower limit for  $\min_{C_0} \max_K J^{2/\infty}(C_0, K, t_{fi})$ .
2. Let us look at the possible outcomes of this algorithm in the limit as  $t_f \rightarrow \infty$ :
  - $J_n^m(t_{fi})$  unbounded and  $J_n^M(t_{fi})$  unbounded or,  $J_n^m(t_{fi})$  converges and  $J_n^M(t_{fi})$  unbounded. Then the  $H_\infty$ -bound is violated and/or the closed-loop system is unstable.
  - $J_n^m(t_{fi})$  converges and  $J_n^M(t_{fi})$  converges but  $J_n^m(t_{fi}) \neq J_n^M(t_{fi})$ . Then the system is asymptotically stable and the  $H_\infty$ -bound is satisfied.
  - $J_n^m(t_{fi})$  converges,  $J_n^M(t_{fi})$  converges and  $J_n^m(t_{fi}) = J_n^M(t_{fi})$ :  
The system is asymptotically stable and the  $H_\infty$ -bound is satisfied and the strategies represent a saddle point if the optimal  $C_0$  and  $K$  are in the interior of their respective domain.

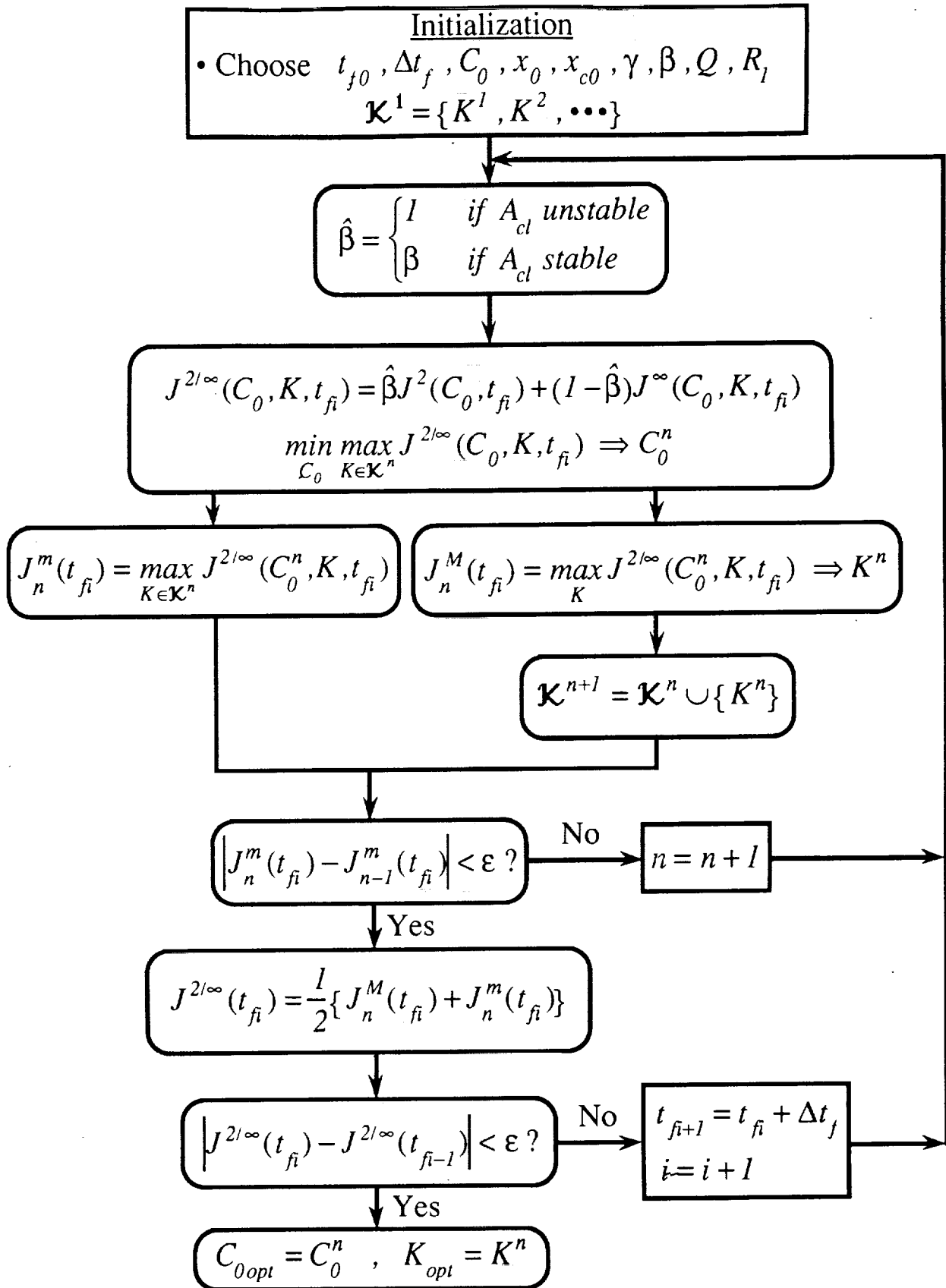


Figure 2: Flow Chart of Design Algorithm

Theoretically there could also be the case of 'limit oscillations' of  $J_n^M(t_{fi})$ , meaning that  $J_n^M(t_{fi})$  does not diverge, but it does not converge to one value either. The system is asymptotically stable and the  $H_\infty$ -bound is satisfied in this case. The actual implementation will show whether this case is relevant and has to be included into the general framework. These possible cases show that the goal is not necessarily the finding of a saddle point but rather we seek the 'boundedness' of  $J_n^m(t_{fi})$  and  $J_n^M(t_{fi})$  in the limit as  $t_f \rightarrow \infty$ . A saddle point strategy is a special case of all the possible strategies that satisfy our design goals.

3. Note, that the algorithm as shown assumes that a stabilizing solution exists, that satisfies the  $H_\infty$ -bound. Other 'checks' have to be incorporated to detect all possible cases as described above. The algorithm as presented shows the schematics and the most important steps only.
4. If the above scheme turns out to be computationally too costly, (Note: all previous computed  $K$  have to be stored) then a sequence of pure maximizations and minimizations (without retaining previous  $K$ ) will be applied. This would be equivalent to finding the worst-case disturbance for an existing system followed by a reoptimization of the controller with respect to these disturbances. This approach has been used for example by Rhee and Speyer in [40].
5. Note that the optimization problem as stated here allows the worst-case  $H_\infty$ -disturbance to be unbounded, if a characterization of the  $H_\infty$ -disturbance  $w$  with  $w \in L_2$  is desired (see e.g. [47]), then this can be incorporated by constraining the maximization steps to an asymptotically stable closed-loop matrix  $A_{cl}$  (with the maximizing  $w$  in place).
6. If the closed-loop system for a certain controller is unstable, then only an  $H_2$ -optimization will be performed until a compensator is found that stabilizes the plant.  $\hat{\beta}$  will perform the selection.
7. The algorithm does not guarantee stability or the  $H_\infty$ -bound for the actual limit as  $t_f \rightarrow \infty$ . Hence, after the design procedure has been terminated at the largest finite time  $t_f$  that can be implemented on the computer, stability has to be checked via the eigenvalues of the closed-loop A-matrix,  $A_{cl}$ , and using the  $H_\infty$ -ARE to examine whether the actual design specifications have been met.
8. The optimization will be embedded into the design tool SANDY and thus uses a gradient-based method to achieve extrema. This however does not necessarily guarantee a global optimum.
9. As the cost function is not convex in  $C_0$  nor is it concave in  $K$ , this approach represents a controller design with worst-case  $H_\infty$ -disturbance achievable by the initial guesses of  $K$  and  $C_0$ .

## 8 Summary and Concluding Remarks

The pure  $H_\infty$ -bound control problem has been solved in algebraic and game theoretical terms. Connections have been developed in the last technical report (see [24]). On the other hand, the mixed  $H_2/H_\infty$ -bound control problem can be viewed as a combination of a stochastic and deterministic game (see e.g. the results of Doyle, Zhou and Bodenheimer in Chapter 4). An algebraic solution cannot be derived as easily as in the pure  $H_\infty$  case. A formulation in terms of a LQ-type cost function is still possible in mixed  $H_2/H_\infty$  control problems where we have to optimize over two different types of disturbances. Solution using standard Lagrange multipliers has to be extended to cover this case. This is a topic of ongoing theoretical research in this field.

A frequency-domain cost as defined in section 6.3 of [24] seems to be promising for the definition of a mixed  $H_2/H_\infty$  cost function. This approach, however, will not be pursued further here. Rather, we will concentrate on the approach as presented in this report. The proposed formulation offers many insights into the mixed  $H_2/H_\infty$ -bound control problem and has many close connections to the algebraic approaches of other researchers as discussed in Chapter 4. A frequency-domain parametrization of the worst-case  $H_\infty$ -disturbance as in section 6.3 of [24] cannot provide these properties.

The approach presented here, uses a parametrization of the worst-case  $H_\infty$ -disturbance in the form of  $w = Kx$  to formulate a cost function that represents a mixed  $H_2/H_\infty$ -bound control strategy for the most general two disturbance/one criterion as well as the one disturbance/two criterion case.

The approach combines a weighted  $H_2$ -criterion with a  $H_\infty$ -bound criterion using a single cost functional. The problem of mixed  $H_2/H_\infty$ -control as posed by the cost function  $J^{2/\infty}(C_0, K)$  furthermore represents a 'simultaneous mixed  $H_2/H_\infty$ -bound-approach'. Extensions to an actual mixed  $H_2/H_\infty$ -design and  $\mu$ -synthesis can be included. The embedded  $H_2$ -problem can be optimized with white noise as driving disturbance or other noise distributions.

The assumed compensator structure is the most general (proper or strictly proper), the controller order can be chosen freely. The initial guess for the compensator need not be stabilizing.

The presented algorithm is well defined for every finite time  $t_f$ , and represents a computational way to solve the min-max problem not in terms of a saddle point solution but in terms of a 'bounded game value' if it exists.

The formulation of the mixed  $H_2/H_\infty$ -control problem as a finite time min-max problem using parameter optimization methods represents a new, non Riccati-based approach in mixed  $H_2/H_\infty$ -bound control.

## 9 Outline of Further Research

1. Implementation of the algorithm and creation of a test-bed for the defined cost function.
2. Test on sample plants and comparison with Riccati-based designs. The comparison will be performed on an analytical basis by comparing the  $K$  that generates the worst-case

$H_\infty$ -disturbance with the solution of the according ARE. Further analysis will include maximum singular value plots and the achieved  $H_2$ -norm.

3. Extension and application of the proposed scheme to the general "two disturbance inputs/two disturbance output"-case. As long as cross-couplings are neglected, this generalization does not pose a significant problem with the defined cost function as we can define two independent systems for the  $H_2$ - and  $H_\infty$ -objective (both systems share the same controller).
4. Simplification of the general system assumptions for a 'two disturbance inputs/two criterion outputs' and relaxation of the imposed constraints (such as orthogonality conditions or observability/controllability conditions). This task will be performed in the spirit of Safonov (see [44] or [66]) by using preliminary state, control and disturbance transformations.

#### Possible Extensions and Long Term Goals:

1. Inclusion of a  $\gamma$ -iteration for an actual minimization of the according  $H_\infty$ -norm.
2. Extension to  $\mu$ -synthesis if the  $\Delta$ -block has a structure.
3. Actual definition of a strategy for the "two disturbance inputs/two disturbance output"-case, taking cross-couplings into account.
4. Modification of the defined cost function to incorporate CLTR-type designs.
5. Possible use of genetic algorithms to solve the posed optimization problem.
6. Extension to a general mixed  $H_X/H_\infty$ -synthesis framework.

#### Work in Progress:

E. Schoemig and Uy-Loi Ly, "Mixed  $H_2/H_\infty$  Control with an Output-Feedback Compensator Using Parameter Optimization," To be presented at the 1992 ACC Conference, Westin Hotel, Chicago, IL, June 24-26, 1992.



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